

# **$T$ -root systems of exceptional flag manifolds. Classification and invariants**

M. M. Graev

ABSTRACT. We classify the systems of  $T$ -roots of the flag manifolds  $M$  of the exceptional compact simple Lie groups with the second Betti number  $b_2(M) \geq 2$ .

## CONTENTS

|   |    |
|---|----|
| List of Tables  | 1  |
| Introduction  | 2  |
| 1. Definitions. Classification of T-root systems of classical types                               | 3  |
| 2. Preliminary  | 4  |
| 2.1. Standard complex forms of flag manifold  | 4  |
| 2.2. Properties of $T$ -root systems  | 6  |
| 2.3. Classification of the exceptional flag manifolds   | 7  |
| 3. Classification of irreducible T-root systems of rank two                                       | 8  |
| 3.1. Detailed classification  | 9  |
| 3.2. Some inclusions between T-root systems   | 12 |
| 4. $T$ -root systems of exceptional flag manifolds with $b_2 \geq 2$                              | 13 |
| 4.1. Flag manifolds with isomorphic $T$ -root systems   | 14 |
| 4.2. Description of $T$ -root systems of ranks $k \geq 2$ with $d \leq 10$                        | 16 |
| 4.3. Exceptional and non-exceptional systems of $T$ -roots  | 16 |
| 5. Tables of T-root systems and the corresponding flag manifolds of exceptional simple Lie groups | 17 |
| 5.1. Some invariants of T-root system   | 17 |
| 5.2. Classification results. Tables   | 18 |
| References  | 27 |

## LIST OF TABLES

|   |   |    |
|---|---|----|
| 1 | Irreducible T-root systems of rank two  | 9  |
| 2 | Exceptional flag manifolds with irreducible $T$ -root systems of rank two                           | 10 |
| 3 | Standard complex forms of exceptional flag manifolds with irreducible $T$ -root systems of rank two | 11 |
| 4 | Classification and invariants of T-root systems of ranks $\geq 2$                                   | 19 |
| 5 | Consistent numberings of simple T-roots   | 22 |
| 6 | The same table as 4 with diagrams of some extremal complex forms                                    | 23 |

---

<sup>0</sup>)Supported by RFBR, grant 13-01-00190.

## INTRODUCTION

A flag manifold of a compact semisimple Lie group  $G$  is an adjoint orbit  $M = \text{Ad}(G).x = G/H$  of  $G$ . It carries many invariant structures, for example, invariant complex structures and associated invariant Kähler-Einstein structures, a big family of invariant Kähler structures, many non-Kähler invariant Einstein metrics etc. An important invariant of a flag manifold is the system of  $T$ -roots. In the case of the full flag manifold  $M = G/T^\ell$  (i.e., a regular orbit of  $G$ ), the system of  $T$ -roots reduces to the system of roots (w.r.t. a maximal torus  $T^\ell \subset G$ ). In the general case the  $T$ -root system can be defined as follows.

Let  $G$  be a compact semisimple Lie group, and  $M = G/H$  a flag manifold. The  $T$ -roots [1] of  $G/H$  are the characters of the torus  $T^k = Z(H)$  corresponding to the one-dimensional  $T^k$ -invariant subspaces of  $\mathbb{C} \otimes (\mathfrak{g}/\mathfrak{h})$ . The system  $\Omega$  of the  $T$ -roots is a subset of the vector space  $\mathbb{R}^k = (\text{Lie}(Z(H)))^*$ . It is easy that  $\Omega$  is not contained in a hyperplane. The number  $k$  is called rank of  $\Omega$ . We consider  $\Omega$  up to linear isomorphism.

In [6], see also § 1, we enumerate the  $T$ -root systems (up to linear isomorphisms) of the flag manifolds  $M$  of the classical simple Lie groups  $G$ . The aim of this paper is to classify the systems of  $T$ -roots of the flag manifolds  $M$  of the exceptional compact simple Lie groups with the second Betti number  $b_2(M) \geq 2$ , that is, with  $k \geq 2$ .

Note that every finite root system is a system of  $T$ -roots. E.g., the root system of type  $G_2$  is the  $T$ -root system of flag manifolds  $G_2/T^2$ ,  $E_6/T^2 \cdot (A_2)^2$ ,  $E_7/T^2 \cdot A_5''$ ,  $E_8/T^2 \cdot E_6$ , and  $F_4/T^2 \cdot \tilde{A}_2$ . This is used in the recent paper [7] to enumerate positive definite invariant Einstein metrics up to isometry and scale on these manifolds (3, 7, 7, 7, 7 metrics respectively, cf. also [8]).

The article is organized as follows:

In § 1 and 2 we state basic definitions and collect some preliminary results.

In § 3 we enumerate irreducible  $T$ -root systems of rank two (Table 1). and the corresponding flag manifolds of Lie groups  $E_6$ ,  $E_7$ ,  $E_8$ , and  $F_4$  (Tables 2 and 3). We prove consequences for the case of rank  $k \geq 3$ .

In § 4 we state main classification results and prove some consequences. In particular, we enumerate flag manifolds of  $E_6$ ,  $E_7$ ,  $E_8$ , and  $F_4$  with isomorphic  $T$ -root systems of ranks  $k \geq 3$  (Theorem 6, cf. Table 5), and describe the  $T$ -root system with  $d \leq 10$  positive  $T$ -roots, corresponding to flag manifolds of  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ .

In § 5 we introduce some simple invariants of  $T$ -root system and get classification tables 4 and 5.

# 1. DEFINITIONS. CLASSIFICATION OF T-ROOT SYSTEMS OF CLASSICAL TYPES

Let  $G$  be a compact connected semisimple Lie group of rank  $\ell$ . We fix a maximal torus  $T^\ell \subset G$ .

A flag manifold  $M = G/H$  of  $G$  is the factor space by the centralizer  $H$  of some torus in  $G$ . Any flag manifold is simply connected. We will suppose later that  $T^\ell \subset H$ . In other words, the basepoint  $eH \in G/H$  belongs to the finite set of the fixed points of  $T^\ell$ .

More precisely (although to fix the idea we identify  $M$  with  $G/H$ ), we will consider  $M$  as a manifold with transitive and locally effective action of the group  $G$  by ignoring the basepoint. In other words, coset spaces  $G/H$  and  $G/H_1$  represent the same flag manifold  $M$  if and only if subgroups  $H$  and  $H_1$  of  $G$  are conjugate,  $H_1 = xHx^{-1}$  for some  $x \in G$ .

We associate now with everyone of considered coset spaces  $G/H$  a finite vector set, called **T-root system** ([1]). With a flag manifold  $M$  one can associate an equivalence class of such systems (up to linear isomorphisms). Later in this section we enumerate  $T$ -root systems (up to isomorphisms) of the flag manifolds of the classical groups (by [6]).

Let  $Q_G$  be the root lattice of  $\mathfrak{g}$  (with respect to  $T^\ell$ ), and  $Q_H \subset Q_G$  the sublattice generated by the roots of  $[\mathfrak{h}, \mathfrak{h}]$ . By definition of the flag manifold  $G/H$ , the quotient group  $Q_G/Q_H$  is torsion-free. More precisely,  $Q_G/Q_H$  can be considered as a vector lattice in  $\mathbb{R}^k$ , the dual space of the Lie algebra of the torus  $T^k = Z(H)$  :

$$\mathbb{R}^k = \mathbb{R} \otimes (Q_G/Q_H) = (\text{Lie } T^k)^*.$$

**Definition.** ([1]) Let  $\alpha$  be a root of  $\mathfrak{g}$ , but not a root of  $[\mathfrak{h}, \mathfrak{h}]$ , that is,  $\alpha \notin Q_H$ . Then the natural projection of  $\alpha$  into  $\mathbb{R}^k \setminus (0)$  (that is,  $\alpha + Q_H$ ) is called a **T-root**.

E.g.,  $T$ -roots of  $G/T^\ell$  are roots of the Lie algebra  $\mathfrak{g}$  with respect to  $T^\ell$ .

We call a **T-root system** of  $G/H$ , or simply a  $T$ -root system, the set  $\Omega = \Omega_{G/H}$  of all  $T$ -roots. The **rank** of  $\Omega$  is the number

$$k = \dim(Z(H)) = b_2(M), \quad \text{the second Betti number of } M.$$

There is a natural correspondence between  $T$ -roots  $\omega \in \Omega$  and irreducible submodules  $\mathfrak{m}_\omega$  of the complex  $H$ -module  $\mathbb{C} \otimes (\mathfrak{g}/\mathfrak{h})$  :

$$\mathfrak{m}_\omega = \bigoplus_{\alpha \in \omega} \mathfrak{g}^\alpha.$$

Indeed, every  $\mathfrak{m}_\omega$  is irreducible by the profound Lemma 3.9 in [5, Chap. 3]. Moreover, submodules  $\mathfrak{m}_\omega$ ,  $\omega \in \Omega$ , are pairwise non-equivalent even as  $Z(H)$ -modules. Since  $\mathfrak{m}_\omega$  and  $\mathfrak{m}_{-\omega}$  are complex conjugate, the *real* isotropy  $H$ -module  $\mathfrak{g}/\mathfrak{h}$  decomposes as a direct sum of

$$d = \frac{1}{2}|\Omega|$$

mutually non-equivalent irreducible submodules.

**Definition.** A **T-root system** in  $\mathbb{R}^k$  is a subset  $A\Omega_{G/H}$  for any  $A \in \text{GL}(k, \mathbb{R})$ . Let  $\text{Aut}(\Omega) := \{A \in \text{GL}(k, \mathbb{R}) : A\Omega = \Omega\}$ .

**Definition.** Two  $T$ -root systems  $\Omega$  and  $\Omega'$  in  $\mathbb{R}^n$  are **isomorphic**, if  $\Omega = A\Omega'$  for some  $A \in \text{GL}(n, \mathbb{R})$ .

Here is an example of such isomorphic systems. If two subgroups  $H_1 = H$  and  $H_2$  of  $G$  are conjugate, i.e.,  $H_2 = xHx^{-1}$  for some  $x \in G$ , then there is a commutative diagram

$$\begin{array}{ccc} Z(H_1) & \xrightarrow{\sim} & Z(H_2) \\ \downarrow & & \downarrow \\ \mathrm{GL}(\mathfrak{g}/\mathfrak{h}_1) & \xrightarrow{\sim} & \mathrm{GL}(\mathfrak{g}/\mathfrak{h}_2) \end{array}$$

Since  $\Omega_{G/H}$  can be defined independently of  $T^\ell$ , starting from the natural action of  $Z(H)$  on  $\mathfrak{g}/\mathfrak{h}$ , this means that  $T$ -root systems  $\Omega_{G/H_i}$ ,  $i = 1, 2$ , are isomorphic.

This observation leads to the following definition.

**Definition.** We call a **T-root system of the flag manifold  $M$**  the isomorphism class  $\Omega_M$  of the  $T$ -root system of  $G/H$ . We will consider later  $\Omega_{G/H}$  up to isomorphism and write  $\Omega_M = \Omega_{G/H}$ .

We enumerate in [6] the  $T$ -root systems of the flag manifolds of the classical groups  $G$ . There are the roots systems of the types  $A_n, B_n, C_n, D_n, BC_n$ , and the systems of vectors obtained from  $C_n$  and  $BC_n$  by removing a part of long roots, as the next theorem states.

**Theorem 1.** *The isomorphism classes of the  $T$ -root systems of the flag manifolds  $M$  of the classical compact simple Lie groups are the following:*

$$\begin{aligned} A_1, \quad BC_1, \quad A_2, \quad B_2, \quad BC_{2,1}, \quad BC_2, \\ A_3, \quad C_{3,1}, \quad C_{3,2}, \quad C_3, \quad B_3, \quad BC_{3,1}, \quad BC_{3,2}, \quad BC_3, \\ A_4, \dots, A_n, \quad D_n, \quad C_{n,1}, \dots, C_{n,n-1}, \quad C_n, \\ B_n, \quad BC_{n,1}, \dots, BC_{n,n-1}, \quad BC_n, \quad A_{n+1}, \dots, \end{aligned}$$

where  $C_{n,n-k}$  and  $BC_{n,n-k}$  respectively are obtained from  $C_n$  and  $BC_n$  by removing a  $k$  pairs of opposite roots of the maximum length. Each of these  $T$ -root systems except  $D_n$ ,  $n > 3$ , corresponds to infinite set of flag manifolds  $M$ , indicated in [6, Introduction].

## 2. PRELIMINARY

**2.1. Standard complex forms of flag manifold.** Assume that  $G^\mathbb{C}$  is the complex semisimple Lie group with the maximal compact subgroup  $G$ . (Here without loss of generality we may assume that  $G = \mathrm{Ad}(G)$ ,  $G^\mathbb{C} = \mathrm{Ad}(G^\mathbb{C})$ ). Fix the standard positive Borel subgroup  $B_+ \subset G^\mathbb{C}$ , so that  $B_+ \cap G = T^\ell$ .

Let  $P$  be a parabolic subgroup of  $G^\mathbb{C}$  such that

$$B_+ \subset P \subset G^\mathbb{C}.$$

Remark that if subgroups  $P$  and  $P'$  are conjugate, and  $B_+ \subset P'$ , then  $P' = P$  (this is an exercise). Moreover, it is known that there are exactly  $2^\ell - 1$  parabolic subgroups  $P \supset B_+$ , labeled by non-empty subsets of the set of simple roots of the Lie algebra  $\mathfrak{g}$ .

**The diagram of  $P$  and of the corresponding parabolic subalgebra  $\mathfrak{p}$ ,**

$$\mathfrak{b}_+ \subset \mathfrak{p} \subset \mathfrak{g}^\mathbb{C}$$

is the Dynkin's diagram of the Lie algebra  $\mathfrak{g}$ , where the vertices (=simple roots) are labeled by  $\{0, 1\}$ . E.g.,  $P = B_+$ , if all simple roots are labeled by 1.

We have the complex manifold  $M = G^\mathbb{C}/P$ . The compact Lie group  $G$  acts transitively on  $M$ . We may consider  $M = G/G \cap P$  as a flag manifold of  $G$  with an  $G$ -invariant complex structure and the standard basepoint  $o = e(G \cap P)$ .

Here is an equivalent infinitesimal construction:

**Definition.** A **standard complex form** of a flag manifold  $M = G/H$  is a pair  $(G/xHx^{-1}, \mathfrak{p})$ , where  $x \in G$ , and  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{g}^\mathbb{C}$  satisfies the following properties:

$$\mathfrak{b}_+ \subset \mathfrak{p}, \quad \mathfrak{p} \cap \mathfrak{g} = \text{Ad}(x)\mathfrak{h}, \quad \text{and hence } T^\ell \subset xHx^{-1}.$$

The **diagram of a standard complex form** is the diagram of  $\mathfrak{p}$ .

Thus, we can associate with a flag manifold  $M = G/H$  with a fixed invariant complex structure the unique standard complex form  $(G/xHx^{-1}, \mathfrak{p})$  of  $M$  and the natural basepoint  $xH \in G/H$  (for the uniqueness, cf. the above remark). We obtain:

**Claim.** *Any complex flag manifold has a unique standard complex form. Any flag manifold has at least one standard complex form.*

Assume for simplicity that  $(G/H, \mathfrak{p})$  is a standard complex form of  $M$  (that is,  $x \in H$ ). We describe explicitly the diagram of  $(G/H, \mathfrak{p})$ . A root  $\alpha$  of the Lie algebra  $\mathfrak{g}$  is positive, if  $\mathfrak{g}^\alpha \in \mathfrak{b}_+$ . In particular let  $\gamma$  be every simple root. Then  $\gamma$  is labeled

- by 1, if  $\mathfrak{g}^\gamma$  belongs to the nilradical of  $\mathfrak{p}$ , that is,  $\mathfrak{g}^\gamma \notin \mathfrak{h}^\mathbb{C}$ ,
- by 0, otherwise.

Given a standard complex form  $(M = G/H, \mathfrak{p})$  we may easily reconstruct the  $T$ -root system  $\Omega = \Omega_{G/H}$  as follows. Assume that  $R$  is the root system of  $\mathfrak{g}$  with respect to the fixed maximal torus  $T^\ell$ . Let  $\{\gamma\}$  be the basis of simple roots, and

$$R_M = \left\{ \sum_{\gamma} k_{\gamma} \gamma \in R \mid \sum_{\gamma \text{ is labeled } 1} |k_{\gamma}| > 0 \right\}.$$

Assume  $\alpha_{\gamma} = \gamma + Q_H$ , cf. §1. Then

$$\Omega = \left\{ \sum_{\gamma \text{ is labeled } 1} k_{\gamma} \alpha_{\gamma} \mid \sum_{\gamma} k_{\gamma} \gamma \in R_M \right\}.$$

Thus, we may associate with a standard complex form  $(M = G/H, \mathfrak{p})$  a basis in  $\mathbb{R}^k$ ,  $k = \text{rank}(\Omega)$ , namely, the basis  $\{\alpha_{\gamma} = \gamma + Q_H : \gamma \text{ is labeled } 1\}$ . This is called **the basis of simple T-roots**.

The set  $\Omega^+$  of **positive T-roots** is the intersection of  $\Omega$  with the closed cone in  $\mathbb{R}^k$  generated by simple  $T$ -roots  $\alpha_i$ ,  $i = 1, \dots, k$ . Obviously, every  $\omega \in \Omega$  is either positive ( $\omega \in \Omega^+$ ), or negative ( $-\omega \in \Omega^+$ ).

**Remark.** The positive Weyl chamber in  $(\mathbb{R}^k)^* = \text{Lie}(Z(H))$  associated with a standard complex form of  $G/H$  is the simplicial cone

$$C = \{h \in \text{Lie}(Z(H)) : \langle h, \alpha_i \rangle > 0, i = 1, \dots, k\}.$$

More generally, Weyl chambers in  $(\mathbb{R}^k)^* = \text{Lie}(Z(H))$  are the connected components of the open set  $\{h : \prod_{\omega \in \Omega_{G/H}} \langle h, \omega \rangle < 0\}$ . Every Weyl chamber is a simplicial cone; there

is a one-to-one correspondence between the set of such cones and the set of the invariant complex structures on  $G/H$  [1]. In this context, a standard complex form represents a class of equivalent invariant complex structures on the flag manifold  $G/H$ .

**2.2. Properties of  $T$ -root systems.** Let  $\Omega$  be a  $T$ -root system of rank  $k$  in  $\mathbb{R}^k$ . A non-empty subset  $\Sigma \subset \Omega$  is

- a closed subsystem, if  $\alpha, \beta \in \Sigma$ ,  $\alpha + \beta \in \Omega$  implies that  $\alpha + \beta \in \Sigma$ ;
- a symmetric subsystem, if  $\alpha \in \Sigma$  implies that  $-\alpha \in \Sigma$ ;
- a complete subsystem, if  $\Sigma = \Omega \cap V$  for some vector subspace  $V \subset \mathbb{R}^k$ .

The system  $\Omega$  is called

- **reducible**, if it can be represented by a union  $\Omega = \Sigma_1 \cup \Sigma_2$  of two complete subsystems  $\Sigma_i = \Omega \cap V_i$ ,  $i = 1, 2$ , with  $V_1 \cap V_2 = (0)$ ,
- **irreducible**, otherwise.

**Lemma 2.1.** *The group  $G$  of a flag manifold  $M$  is simple if and only if the  $T$ -root system  $\Omega_M$  is irreducible.*

**Claim.** *Every closed symmetric subsystem  $\Sigma \subset \Omega$  is a  $T$ -root system of rank*

$$\text{rank}(\Sigma) = \dim(\text{span}(\Sigma)), \quad \text{rank}(\Sigma) \leq \text{rank}(\Omega),$$

*in the vector space  $\text{span}(\Sigma)$  spanned by  $\Sigma$ .*

Claim follows immediately from the next Lemma 2.2 that we state without proof.

Assume that  $L$  is a compact connected semisimple Lie group and  $N = L/H$  is a flag manifold such that

$$\Omega_N = \Omega.$$

**Lemma 2.2.** *Let  $\Omega = \Omega_N$  be  $T$ -root system of a flag manifold  $N = L/H$ , and  $\Sigma \subset \Omega$  a closed symmetric subsystem. Then there exists a  $H$ -invariant connected semisimple subgroup  $G \subset L$  such that  $M = G/G \cap H$  is a flag manifold with the  $T$ -root system*

$$\Omega_M = \Sigma$$

*of rank  $\text{rank}(\Omega_M) = \dim(\text{span}(\Sigma))$ . Moreover, if  $\Sigma$  is irreducible, then the group  $G$  of  $M = G/G \cap H$  is simple.*

The last assertion follows from Lemma 2.1.

Note that  $G \cap H$  is a normal subgroup of the group  $H$ . This implies:

**Corollary.** *If  $\text{rank}(L) - \text{rank}(G) = 1$ , then  $[H, H] \subset G \cap H$ .*

**Lemma 2.3.** *Let  $\Omega$  be an irreducible system of  $T$ -roots of rank  $k$ , and  $\Sigma_r$  a complete irreducible subsystem of rank  $r < k$ . Then there exists a complete irreducible subsystem  $\Sigma_{r+1}$  of rank  $r + 1$  such that*

$$\Sigma_r \subset \Sigma_{r+1} \subset \Omega.$$

**2.3. Classification of the exceptional flag manifolds.** The classification of exceptional flag manifolds is, of course, very known. It is useful to reading the present paper. Note that our calculations are **independent** of this classification. But some checks must be simplified, if we use the following proposition.

**Proposition 2.1.** *Let  $M = G/H$  be a flag manifold with  $b_2(M) > 1$  of an exceptional compact Lie group  $G$  of type  $E_\ell$ ,  $\ell = 6, 7, 8$ . Assume that  $H$  contains the fixed maximal torus  $T^\ell$  of  $G$ . Then up the natural action of the corresponding Weyl group  $W = \text{Norm}_G(T^\ell)/T^\ell$ , the group  $H$  is of one and only one of the following kinds:*

- 1)  $H$  contains a normal subgroup of type  $E_6$  or  $D_k$ ,  $k > 3$  (there is 1, 3, 8 such manifolds  $G/H$  respectively),
- 2)  $[H, H]$  belongs to the standard  $A_{\ell-1}$  (there is  $p(\ell) - 1 = 10, 14, 21$  such manifolds, where  $p(n)$  is the number of partitions of  $n$ ),
- 3)  $H$  can be described by one of the following diagrams:

$$\begin{array}{c} 0 \ 1 \ 0 \ 1 \ 0 \\ 0 \end{array} E_6/T^2 \cdot A_1 \cdot A_1 \cdot A_2,$$

$$\begin{array}{c} 1 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \end{array} E_7/T^2 \cdot A_1 \cdot A_2 \cdot A_2,$$

$$\begin{array}{c} 0 \ 1 \ 0 \ 1 \ 0 \ 0 \\ 0 \end{array} E_7/T^2 \cdot A_1 \cdot A_1 \cdot A_1 \cdot A_2,$$

$$\begin{array}{c} 0 \ 0 \ 0 \ 1 \ 1 \ 0 \\ 0 \end{array} E_7/T^2 \cdot A_1 \cdot A_1 \cdot A_3,$$

$$\begin{array}{c} 0 \ 0 \ 0 \ 0 \ 1 \ 1 \\ 0 \end{array} E_7/T^2 \cdot [A_5]'',$$

$$\begin{array}{c} 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\ 0 \end{array} E_7/T^3 \cdot A_1 \cdot A_1 \cdot A_1 \cdot A_1,$$

$$\begin{array}{c} 0 \ 0 \ 0 \ 1 \ 1 \ 1 \\ 0 \end{array} E_7/T^3 \cdot [A_1 \cdot A_3]'',$$

$$\begin{array}{c} 0 \ 1 \ 0 \ 1 \ 1 \ 1 \\ 0 \end{array} E_7/T^4 \cdot [A_1 \cdot A_1 \cdot A_1]'',$$

$$\begin{array}{c} 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \\ 0 \end{array} E_8/T^2 \cdot A_1 \cdot A_2 \cdot A_3,$$

$$\begin{array}{c} 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \\ 0 \end{array} E_8/T^3 \cdot A_1 \cdot A_1 \cdot A_1 \cdot A_2,$$

$$\begin{array}{c} 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \\ 0 \end{array} E_8/T^2 \cdot A_1 \cdot A_1 \cdot A_2 \cdot A_2,$$

$$\begin{array}{c} 0 \ 1 \ 0 \ 0 \ 1 \ 0 \\ 0 \end{array} E_8/T^2 \cdot A_1 \cdot A_1 \cdot A_4.$$

Remark that in general a flag manifold of the kind 3) has more than one standard complex forms.

*Proof for  $G = E_8$ .* We prove the case 3) using the combinatorics of Dynkin's diagrams. Let  $H$  is not of the kinds 1), 2). Then it can be described by one of the diagrams:

$$a) \begin{array}{c} x \ x \ x \ 1 \ x \ x \ x \\ 0 \end{array}, \quad b) \begin{array}{c} x \ x \ x \ 0 \ 0 \ 1 \ x \\ 0 \end{array}, \quad c) \begin{array}{c} x \ x \ x \ 0 \ 1 \ x \ x \\ 0 \end{array},$$

where each  $x$  must be replaced by 0 or 1.

a) The condition a) implies  $\begin{array}{c} x \ x \ x \ 1 \ x \ 0 \ 0 \\ 0 \end{array}$  (otherwise  $H$  reduces to 2)) Then  $\begin{array}{c} x \ x \ 0 \ 1 \ x \ 0 \ 0 \\ 0 \end{array}$ ; otherwise we reduce  $H$  to the case 2) using  $-w \in W(E_8)$ , where  $-1_8$  is an element of

the Weyl group  $W(E_8)$ , and  $w$  is the element of the maximum length of an appropriate subgroup  $W(D_5)$ . Hence, we have  ${}^0 x {}^0 1 {}^x 0 {}^0$ . Because  $b_2(M) > 1$ , we obtain the diagrams

$${}^0 0 {}^0 0 {}^1 1 {}^0 0 {}^0, \quad {}^0 1 {}^0 1 {}^1 1 {}^0 0 {}^0, \quad {}^0 1 {}^0 1 {}^0 0 {}^0 {}^0$$

of the flag manifolds  $E_8/T^2 \cdot A_1 \cdot A_2 \cdot A_3$ ,  $E_8/T^3 \cdot A_1 \cdot A_1 \cdot A_1 \cdot A_2$ ,  $E_8/T^2 \cdot A_1 \cdot A_1 \cdot A_4$ .

c) The condition c) implies  ${}^0 0 {}^x 0 {}^1 x {}^x$  (otherwise  $H$  reduces to  $a$ )), and hence  ${}^0 0 {}^x 0 {}^1 0 {}^0$  (otherwise  $H$  reduces to  $a$ ) or  $b$ )). Since  $b_2(M) \geq 2$ , we obtain the diagram  ${}^0 0 {}^1 0 {}^1 0 {}^0$  of  $E_8/T^2 \cdot A_1 \cdot A_1 \cdot A_2 \cdot A_2$ .

b) We prove that  $b$ ) reduces to  $a$ ). The condition  $b$ ) implies  ${}^0 x {}^x 0 {}^0 1 {}^x$  (otherwise  $H$  reduces to 2)), and  ${}^0 x {}^0 0 {}^0 1 {}^x$  (otherwise  $H$  reduces to  $a$ )). Then  ${}^0 x {}^0 0 {}^0 1 {}^0$ ; otherwise we reduce  $H$  to the case 2) using  $-w \in W(E_8)$ , where  $w$  is the element of the maximum length of an appropriate subgroup  $W(D_7)$  (this is correct, since  $D_7$  is of an odd rank). Because  $b_2(M) > 1$ , we obtain the diagram  ${}^0 1 {}^0 0 {}^0 1 {}^0$  of the flag manifold  $E_8/T^2 \cdot A_1 \cdot A_1 \cdot A_4$ . Let  $u$  be the element of the maximum length of  $W(E_6)$ . Then

$$-u : {}^0 1 {}^0 0 {}^0 1 {}^0 \longrightarrow {}^0 1 {}^0 1 {}^0 0 {}^0,$$

where the right diagram is of the kind  $a$ ). □

**Corollary.** *There exist 77 flag manifolds  $M$  with  $b_2(M) \geq 2$  of the exceptional compact simple Lie groups, namely:*

- 33 = 8 + 21 + 4 flag manifolds of  $E_8$ ,
- 24 = 3 + 14 + 7 flag manifolds of  $E_7$ ,
- 12 = 1 + 10 + 1 flag manifolds of  $E_6$ ,
- 7 flag manifolds of  $F_4$ ,
- 1 flag manifold of  $G_2$ .

### 3. CLASSIFICATION OF IRREDUCIBLE T-ROOT SYSTEMS OF RANK TWO

We consider a flag manifold  $M = G/H$  as a  $G$ -manifold (obtained by ignoring the basepoint  $eH$ ). The isomorphism class  $\Omega_M$  of the  $T$ -root systems  $\Omega_{G/xHx^{-1}}$  is well-defined (independent of the choice of basepoints  $xH$ ), and we call  $\Omega_M$  the  $T$ -root system of the flag manifold  $M$ , cf. §1.

Any flag manifold  $M$  has finitely many standard complex forms. Standard complex forms of  $M$  are defined in §2.1.

**Theorem 2.** *There exist*

- 16 irreducible  $T$ -root systems  $\Omega$  of rank two (up to isomorphism),
- 30 flag manifolds  $M$  with  $b_2(M) = 2$  of the Lie groups  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$ , and  $G_2$ ,
- 71 standard complex forms of such manifolds.

*The systems  $\Omega$  are pictured in Table 1. The correspondence between systems  $\Omega$ , flag manifolds  $M$  and standard complex forms of  $M$  is given by Tables 2 and 3.*



**Theorem 3.** *Let  $\Omega$  be an irreducible  $T$ -root system of rank two. Then there exists at least one exceptional compact simple Lie group  $L$  other than  $G_2$ , and a unique flag manifold  $M$  of  $L$  with a  $T$ -root system isomorphic to  $\Omega$ .*

**3.1. Detailed classification.** Fix an irreducible  $T$ -root system  $\Omega$  of rank two. Let  $\text{Conv}(\Omega) \subset \mathbb{R}^2$  be its convex hull, and let  $P \subset \mathbb{R}^2$  be a fundamental parallelogram of the vector lattice generated by the elements of  $\Omega$ .

**Proposition 3.1.**  *$\Omega$  is uniquely determined by two numbers*

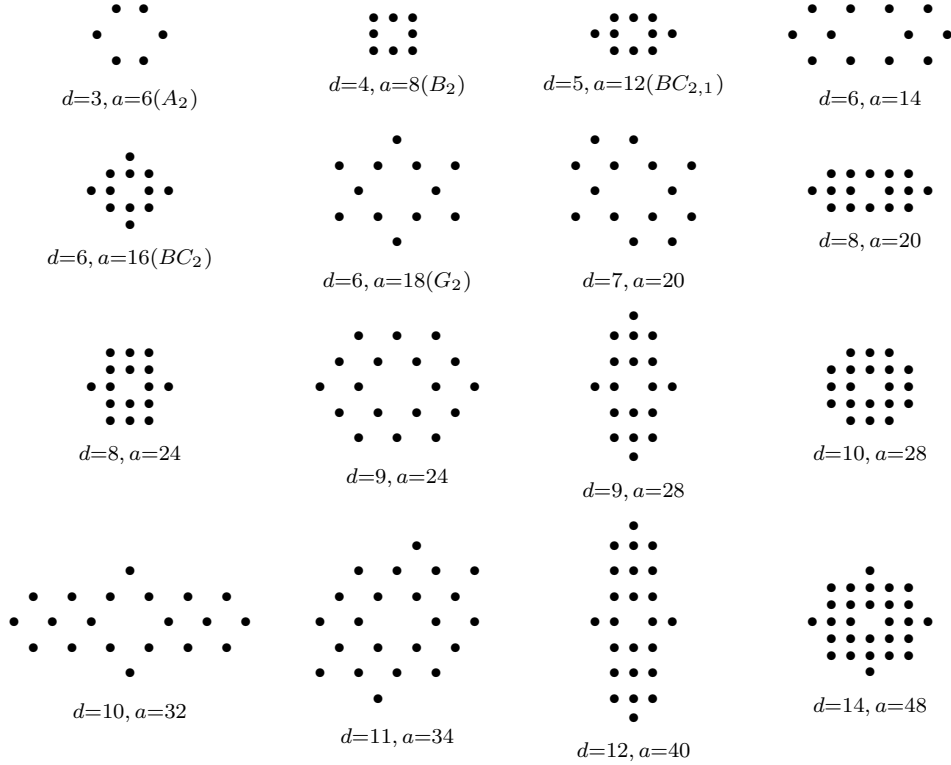
$$d = \frac{1}{2}|\Omega|, \quad a = \frac{2 \text{area}(\text{Conv}(\Omega))}{\text{area}(P)}.$$

**Proposition 3.2.** *Let  $L$  be a compact Lie group of a type  $E_6$ ,  $E_7$ ,  $E_8$ , or  $F_4$ , and let  $(L/H_i, \mathfrak{p}_i)$ ,  $i = 1, 2$ , are two standard complex flag manifolds of  $L$ . If the  $T$ -root systems corresponding to  $\mathfrak{p}_i$  are isomorphic to  $\Omega$ , then the subgroups  $H_i$ ,  $i = 1, 2$  are conjugate.*

Hence, there is at most one flag manifold of  $L$  with the  $T$ -root system  $\Omega$ .

Propositon 3.2 and Theorem 3 follows from Table 3.

TABLE 1. Irreducible  $T$ -root systems of rank two



$d = \frac{1}{2}|\Omega|$ ;  $a$  : the normalized area of the convex hull of  $\Omega$  (Proposition 3.2)

TABLE 2. Exceptional flag manifolds with irreducible  $T$ -root systems of rank two

We enumerate the flag manifolds with  $T$ -root systems  $\Omega$  of rank 2 of the compact Lie groups of types  $E_\ell$ ,  $\ell = 6, 7, 8$ , and  $F_4$ . Notations:  $n_-^o$  : the number of picture (Table 1);

$d = \frac{1}{2}|\Omega|$ ;

$a$  : the normalized area of the convex hull of  $\Omega$  (Proposition 3.2);

$\text{type}(\Omega)$  : the type of  $\Omega$ , e.g., the root system of the type  $G_2$  or  $BC_2$  (cf. Theorem 1);

$m = \max\{k : k\omega \in \Omega, \omega \in \Omega\}$ ;

$f$  : the number of the standard complex forms of a flag manifold.

We use Dynkin's notations [3, §3 (17)]:  $A'_5, A''_5 \subset E_7$ , and  $A_k, \tilde{A}_k \subset F_4$ , where  $k \in \{1, 2\}$ .

| $n_-^o$ | $d$ | $a$ | $\text{type}(\Omega)$ | $m$ | $E_6/T^2 \bullet$ | $f$ | $E_7/T^2 \bullet$ | $f$ | $E_8/T^2 \bullet$ | $f$ | $F_4/T^2 \bullet$ | $f$ |
|---------|-----|-----|-----------------------|-----|-------------------|-----|-------------------|-----|-------------------|-----|-------------------|-----|
| 16      | 14  | 48  |                       | 3   |                   |     |                   |     | $A_1 A_1 A_2 A_2$ | 2   |                   |     |
| 15      | 12  | 40  |                       | 4   |                   |     |                   |     | $A_1 A_2 A_3$     | 4   |                   |     |
| 14      | 11  | 34  |                       | 2   |                   |     |                   |     | $A_1 A_1 A_4$     | 4   |                   |     |
| 13      | 10  | 32  |                       | 3   |                   |     |                   |     | $A_2 A_4$         | 4   |                   |     |
| 12      | 10  | 28  |                       | 2   |                   |     |                   |     | $A_3 A_3$         | 2   |                   |     |
| 11      | 9   | 28  |                       | 3   |                   |     |                   |     | $A_1 A_5$         | 3   |                   |     |
| 10      | 9   | 24  |                       | 2   |                   |     | $A_1 A_1 A_1 A_2$ | 1   | $A_2 D_4$         | 1   | $A_2$             | 1   |
| 9       | 8   | 24  |                       | 2   |                   |     | $A_1 A_1 A_3$     | 3   | $A_1 D_5$         | 3   | $A_1 \tilde{A}_1$ | 3   |
| 8       | 8   | 20  |                       | 3   |                   |     | $A_1 A_2 A_2$     | 3   |                   |     |                   |     |
| 7       | 7   | 20  |                       | 2   |                   |     | $A_2 A_3$         | 3   | $A_6$             | 3   |                   |     |
| 6       | 6   | 18  | $G_2$                 | 1   | $A_2 A_2$         | 1   | $A''_5$           | 1   | $E_6$             | 1   | $\tilde{A}_2$     | 1   |
| 5       | 6   | 16  | $BC_2$                | 2   |                   |     | $A_1 D_4$         | 1   | $D_6$             | 1   | $B_2$             | 1   |
| 4       | 6   | 14  |                       | 2   | $A_1 A_1 A_2$     | 5   | $A_1 A_4$         | 5   |                   |     |                   |     |
| 3       | 5   | 12  | $BC_{2,1}$            | 2   | $A_1 A_3$         | 4   | $A'_5$            | 2   |                   |     |                   |     |
| 2       | 4   | 8   | $B_2$                 | 1   | $A_4$             | 4   | $D_5$             | 2   |                   |     |                   |     |
| 1       | 3   | 6   | $A_2$                 | 1   | $D_4$             | 1   |                   |     |                   |     |                   |     |
| всего:  |     |     |                       |     | 5                 | 15  | 9                 | 21  | 11                | 28  | 4                 | 6   |

16 systems of  $T$ -roots,  $30 = 5 + 9 + 11 + 4 + 1$  flag manifolds (together with  $G_2/T^2$ ),  
 $71 = 15 + 21 + 28 + 6 + 1$  standard complex forms.

TABLE 3. Standard complex forms of exceptional flag manifolds with irreducible  $T$ -root systems of rank two

We enumerate the standard complex forms of the flag manifolds with  $T$ -root systems  $\Omega$  of rank 2 of the compact Lie groups  $E_6$ ,  $E_7$ ,  $E_8$ , and  $F_4$ .

| $d$ | $a$ | $E_6$   | $E_7$   | $E_8$   | $F_4$   |
|-----|-----|---|---|---|---|
| 9   | 28  |   |   | $\begin{array}{c} 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 1 \\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ 1 \\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0 \end{array}$ |   |
| 9   | 24  |   | $\begin{array}{c} 0\ 1\ 0\ 1\ 0\ 0 \\ 0 \end{array}$  | $\begin{array}{c} 0\ 0\ 1\ 0\ 0\ 0\ 1 \\ 0 \end{array}$   | $1\ 1 \Leftarrow 0\ 0$  |
| 8   | 24  |   | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 1 \\ 0 \\ 0\ 0\ 0\ 1\ 1\ 0 \\ 0 \\ 0\ 1\ 0\ 0\ 1\ 0 \\ 0 \end{array}$  | $\begin{array}{c} 1\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0 \\ 0\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0 \\ 0\ 1\ 0\ 0\ 0\ 0\ 1 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 1 \Leftarrow 0\ 1 \\ 0\ 1 \Leftarrow 1\ 0 \\ 1\ 0 \Leftarrow 1\ 0 \end{array}$ |
| 8   | 20  |   | $\begin{array}{c} 0\ 0\ 1\ 0\ 1\ 0 \\ 0 \\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0 \\ 1\ 0\ 0\ 1\ 0\ 0 \\ 0 \end{array}$  |   |   |
| 7   | 20  |   | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 0 \\ 1 \\ 0\ 0\ 1\ 0\ 0\ 0 \\ 1 \\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0 \end{array}$  | $\begin{array}{c} 0\ 0\ 0\ 0\ 0\ 1\ 1 \\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\ 1 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1 \end{array}$ |   |
| 6   | 18  | $\begin{array}{c} 0\ 0\ 1\ 0\ 0 \\ 1 \end{array}$   | $\begin{array}{c} 0\ 0\ 0\ 0\ 1\ 1 \\ 0 \end{array}$  | $\begin{array}{c} 1\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0 \end{array}$   | $0\ 0 \Leftarrow 1\ 1$  |
| 6   | 16  |   | $\begin{array}{c} 0\ 1\ 0\ 0\ 0\ 1 \\ 0 \end{array}$  | $\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 0\ 1 \\ 0 \end{array}$   | $1\ 0 \Leftarrow 0\ 1$  |
| 6   | 14  | $\begin{array}{c} 0\ 0\ 1\ 0\ 1 \\ 0 \\ 0\ 0\ 1\ 1\ 0 \\ 0 \\ 0\ 1\ 0\ 1\ 0 \\ 0 \\ 0\ 1\ 1\ 0\ 0 \\ 0 \\ 1\ 0\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 1\ 0\ 0\ 0 \\ 1 \\ 0\ 0\ 0\ 0\ 1\ 0 \\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0 \\ 1\ 0\ 1\ 0\ 0\ 0 \\ 0 \\ 0\ 1\ 1\ 0\ 0\ 0 \\ 0 \end{array}$ |   |   |
| 5   | 12  | $\begin{array}{cc} 0\ 0\ 0\ 1\ 0 & 0\ 1\ 0\ 0\ 0 \\ 1 & 1 \\ 1\ 0\ 0\ 1\ 0 & 0\ 1\ 0\ 0\ 1 \\ 0 & 0 \end{array}$                          | $\begin{array}{c} 0\ 0\ 0\ 0\ 0\ 1 \\ 1 \\ 1\ 0\ 0\ 0\ 0\ 0 \\ 1 \end{array}$   |   |   |
| 4   | 8   | $\begin{array}{cc} 1\ 1\ 0\ 0\ 0 & 0\ 0\ 0\ 1\ 1 \\ 0 & 0 \\ 1\ 0\ 0\ 0\ 0 & 0\ 0\ 0\ 0\ 1 \\ 1 & 1 \end{array}$                          | $\begin{array}{c} 1\ 0\ 0\ 0\ 0\ 1 \\ 0 \\ 1\ 1\ 0\ 0\ 0\ 0 \\ 0 \end{array}$   |   |   |
| 3   | 6   | $\begin{array}{c} 1\ 0\ 0\ 0\ 1 \\ 0 \end{array}$   |   |   |   |

| $d$ | $a$ | $E_8$   |
|-----|-----|---|
| 14  | 48  | $\begin{array}{c} 0\ 0\ 1\ 0\ 1\ 0\ 0 \\ 0 \\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \\ 0 \end{array}$   |
| 12  | 40  | $\begin{array}{c} 0\ 0\ 1\ 0\ 0\ 1\ 0 \\ 0 \\ 0\ 0\ 0\ 1\ 0\ 1\ 0 \\ 0 \\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \\ 0 \end{array}$ |
| 11  | 34  | $\begin{array}{c} 0\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0 \\ 0\ 0\ 0\ 0\ 1\ 1\ 0 \\ 0 \\ 0\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0 \\ 0\ 1\ 0\ 1\ 0\ 0\ 0 \\ 0 \end{array}$ |
| 10  | 32  | $\begin{array}{c} 0\ 0\ 0\ 0\ 1\ 0\ 0 \\ 1 \\ 0\ 0\ 1\ 0\ 0\ 0\ 0 \\ 1 \\ 0\ 0\ 1\ 1\ 0\ 0\ 0 \\ 0 \\ 1\ 0\ 0\ 1\ 0\ 0\ 0 \\ 0 \end{array}$ |
| 10  | 28  | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ 1 \\ 0\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0 \end{array}$   |

**3.2. Some inclusions between T-root systems.** Consider consequences of the above classification of the irreducible  $T$ -root systems of rank two.

**Proposition 3.3.** *Let  $\Sigma$  and  $\Omega$  are irreducible  $T$ -root systems,  $\Sigma \subset \Omega$ ,  $n = \text{rank}(\Omega) \geq 3$ ,  $\text{rank}(\Sigma) = 2$ ,  $d = \frac{1}{2}|\Sigma| \geq 7$ . Assume  $\Sigma \subset \Omega$  is a symmetric closed subsystem which generates a plane in  $\mathbb{R}^n$ . Let  $M$  and  $N$  are flag manifolds of compact Lie groups  $G$  and  $L$  such that  $\Omega_M = \Sigma$  and  $\Omega_N = \Omega$ . Then*

$$M = E_7/T^2 \cdot H, \quad N = E_8/T^3 \cdot H$$

for one of the following subgroup  $H$  of rank 5 :

$$A_1 \cdot A_1 \cdot A_1 \cdot A_2, \quad A_1 \cdot A_2 \cdot A_2, \quad A_1 \cdot A_1 \cdot A_3, \quad A_2 \cdot A_3.$$

The flag manifolds  $M$  and  $N$  depend only of  $\Sigma$ .

*Proof.* By Lemma 2.3 there is an irreducible  $T$ -root system  $\Sigma_3$  of rank 3 such that  $\Sigma \subset \Sigma_3 \subset \Omega$ . Further,  $G$  is an exceptional simple Lie group other than  $G_2$ , since  $\Sigma$  is irreducible and  $d \geq 7$ . From Table 2 it follows that either  $M = E_7/T^2 \cdot H$ , or  $G = E_8$ . By Lemma 2.2,  $G = E_7$ ,  $L = E_8$ ,  $\Omega = \Sigma_3$ , and  $N = E_8/T^3 \cdot H$ . Then the flag manifold  $N$  has one of the following standard complex forms:

| $A_1 A_1 A_1 A_2$                                       | $A_1 A_2 A_2$   | $A_1 A_1 A_3$   | $A_2 A_3$   |
|---|---|---|---|
| $\begin{array}{c} 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 1\ 0\ 0\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 1\ 0\ 0\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 1\ 0\ 0\ 1 \\ 0 \end{array}$ |
| $\begin{array}{c} 0\ 1\ 1\ 0\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 0\ 1\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 1\ 0\ 0\ 0 \\ 1 \end{array}$ |
| $\begin{array}{c} 0\ 1\ 0\ 1\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 1\ 0\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 0\ 1\ 0\ 1 \\ 0 \end{array}$ | $\begin{array}{c} 1\ 0\ 0\ 0\ 1\ 0\ 0 \\ 1 \end{array}$ |
| $\begin{array}{c} 0\ 1\ 0\ 1\ 0\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 1\ 0\ 0\ 1\ 0\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 1\ 1\ 0\ 0\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 1\ 0\ 0\ 1 \\ 0 \end{array}$ |
| $\begin{array}{c} 0\ 1\ 0\ 0\ 1\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 1\ 1\ 0\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 1\ 0\ 1\ 0\ 0\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 0\ 0\ 1\ 1 \\ 0 \end{array}$ |
| $\begin{array}{c} 0\ 1\ 0\ 0\ 1\ 0\ 1 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 1\ 0\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 1\ 0\ 0\ 0\ 1\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 0\ 0\ 0\ 1 \\ 1 \end{array}$ |
| $\begin{array}{c} 0\ 0\ 1\ 0\ 1\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 0\ 1\ 0\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 0\ 0\ 1\ 1\ 1\ 0 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 1 \end{array}$ |
| $\begin{array}{c} 0\ 0\ 1\ 0\ 1\ 0\ 1 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 1\ 0\ 0\ 1\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 0\ 0\ 1\ 1\ 0\ 1 \\ 0 \end{array}$ | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 1\ 1 \\ 0 \end{array}$ |
|   |   | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 1\ 0 \\ 1 \end{array}$ | $\begin{array}{c} 0\ 0\ 0\ 1\ 0\ 0\ 1 \\ 1 \end{array}$ |

It is easy that the diagrams in each column defines complex forms of the same flag manifold, and hence  $N$  depends only of  $\Sigma$ .  $\square$

Assume now  $\Sigma$  is an irreducible  $T$ -root system of rank two satisfying Proposition 3.3, and  $\alpha \in \Sigma$  is a  $T$ -root such that

$$3\alpha \in \Sigma.$$

From Table 2 it follows that  $\Sigma$  is the root system of the flag manifold  $E_7/T^2 \cdot A_1 \cdot A_2 \cdot A_2$ . Thus, we obtain:

**Proposition 3.4.**  *$E_8/T^3 \cdot A_1 \cdot A_2 \cdot A_2$  is the unique flag manifold  $G/H$  with an irreducible  $T$ -root system  $\Omega$  of rank  $\geq 3$  such that  $\alpha$  and  $3\alpha$  are both  $T$ -roots for some  $\alpha \in \Omega$ .*

*Proof.* By Lemma 2.3 we have  $\alpha, 3\alpha \in \Sigma \subset \Omega$  for some complete irreducible subsystem  $\Sigma$  with  $\text{rank}(\Sigma) = 2$ . From Table 2 it follows that  $d = \frac{1}{2}|\Sigma| \geq 8$ , so  $\Sigma$  satisfies Proposition 3.3. The assertion follows.  $\square$

*A direct proof ( $G = E_8$ ).* The left part of the following table contains the diagrams of the all mutually non-equivalent invariant complex structures on the flag manifold  $E_8/T^3 \cdot A_1 \cdot A_2 \cdot A_2$ . The right part of this table contains the all positive roots of the Lie algebra  $E_8$  with at least 3 coefficients  $k_{i_j} = 0 \pmod{3}$ ,  $i_j \in [1, 8]$ , such that  $k_{i_1} > 0$  (there exist 16 such roots).

| diagram  | root 1   | root 2   |
|--|--|--|
| $\begin{smallmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & 0 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 0 & 1 & 2 & 3 & 2 & 1 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 0 & 1 & 2 & 3 & 2 & 1 \\ & & & & & & 2 \end{smallmatrix}$ |
| $\begin{smallmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ & & & & & & 0 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ & & & & & & 2 \end{smallmatrix}$ |
| $\begin{smallmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ & & & & & & 0 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 1 \\ & & & & & & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 2 \\ & & & & & & 2 \end{smallmatrix}$ |
| $\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ & & & & & & 0 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 3 & 3 & 2 & 1 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 3 & 3 & 2 & 1 \\ & & & & & & 2 \end{smallmatrix}$ |
| $\begin{smallmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ & & & & & & 0 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 3 & 4 & 3 & 1 \\ & & & & & & 2 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 3 & 4 & 3 & 2 \\ & & & & & & 2 \end{smallmatrix}$ |
| $\begin{smallmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 3 & 1 \\ & & & & & & 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 3 & 2 \\ & & & & & & 3 \end{smallmatrix}$ |
| $\begin{smallmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 4 & 6 & 4 & 2 \\ & & & & & & 3 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 2 & 3 & 5 & 6 & 4 & 2 \\ & & & & & & 3 \end{smallmatrix}$ |
| $\begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & 1 \end{smallmatrix}$ | $\begin{smallmatrix} 1 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & & 3 \end{smallmatrix}$ | $\begin{smallmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & & 3 \end{smallmatrix}$ |

(see, e.g., tables in [2]). This table prove Proposition 3.4 for  $G = E_8$ .  $\square$

**Corollary.** *Let  $\Omega$  be an irreducible  $T$ -root system of rank  $\geq 3$ , and  $\omega \in \Omega$  a  $T$ -root. If  $\frac{1}{k}\omega \in \Omega$  then either  $k \in \{\pm 1, \pm 2\}$ , or  $k \in \{\pm 3\}$  and  $\Omega$  is the  $T$ -root system of rank 3, corresponding to the flag manifold  $E_8/T^3 \cdot A_1 \cdot A_2 \cdot A_2$ .*

#### 4. $T$ -ROOT SYSTEMS OF EXCEPTIONAL FLAG MANIFOLDS WITH $b_2 \geq 2$

In this paper we classify the  $T$ -root systems of ranks  $\geq 2$  corresponding to flag manifolds of exceptional simple Lie groups. We state main results of the classification.

**Theorem 4.** *Let  $L$  be an exceptional compact simple Lie group,  $L/H_i$ ,  $i = 1, 2$  are flag manifolds with second Betti numbers  $b_2(L/H_i) = k$ , and  $\Omega_i$  the corresponding  $T$ -root systems of rank  $k$ . If  $k \geq 2$ , then the following conditions are equivalent:*

- 1)  $\Omega_1$  and  $\Omega_2$  are isomorphic;
- 2)  $H_1$  and  $H_2$  are conjugate in  $L$ .

*Sketch of a Proof.* Let  $\Sigma$  be the set of Dynkin diagrams of standard complex forms of the flag manifolds  $M$  of the group  $L$  with  $b_2 > 1$ . Each vertex of  $S \in \Sigma$  is labelled 0 or 1.

1) Given every diagram  $S \in \Sigma$ , we calculate the corresponding  $T$ -root system  $\Omega = \Omega(S)$  (as in § 2.1), and four numeric invariants  $k, d, c, v$  of  $\Omega$ , where  $k = \text{rank}(\Omega)$ ,  $d = \frac{1}{2}|\Omega|$ , and  $c, v$  are defined in § 5 below.

2) One can check directly that the diagrams  $S$  with fixed  $k, d, c, v$  correspond to conjugated flag manifolds of  $L$  (under the group of inner automorphisms of  $L$ ).

Remark that 1) can be easily programmed.

(Remark 2. For another proof we may use Proposition 2.1 and associate with each  $M$  a standard complex form  $S_M \in \Sigma$ . Then it is sufficient to check that  $S_M$  is uniquely determined by  $k, d, c, v$ .)  $\square$

**Theorem 5.** *The classification of systems  $\Omega$  of  $T$ -roots of the flag manifolds  $M$  with  $b_2(M) \equiv \text{rank}(\Omega) \geq 2$  of groups  $E_6$ ,  $E_7$ ,  $E_8$ , and  $F_4$  is given by Table 4 in § 5 below. We skip the group  $G_2$ . The  $T$ -roots of  $M = G_2/T^2$  coincides, obviously, with roots of  $G_2$ .*

Each line of tables corresponds to a  $T$ -root system (up to isomorphism).

Now, we give some consequences of such classification.

**4.1. Flag manifolds with isomorphic  $T$ -root systems.** By definition of a flag manifold  $M = G/H$  we ignore the basepoint  $eH$ , and consider  $M$  with the basepoints  $xH$ . Recall that the  $T$ -root system  $\Omega$  of a coset space  $G/xHx^{-1}$  is unique, up to isomorphism (cf. §1).

**Definition.** Given an irreducible  $T$ -root system  $\Omega$ , denote by  $\mathcal{F}(\Omega)$  the set of the flag manifolds with  $T$ -root systems isomorphic to  $\Omega$ .

**Theorem 6.** *Let  $\mathcal{F}(\Omega)$  contains an exceptional flag manifold,  $|\mathcal{F}(\Omega)| \geq 2$ , and  $\text{rank}(\Omega) \geq 3$ . Then  $\mathcal{F}(\Omega)$  is one of the followings<sup>1)</sup>:*

- |    |  |                                      |                        |  |
|----|--|--------------------------------------|------------------------|--|
| 1) | $E_7/T^4[A_1A_1A_1]''$ ,               | $E_8/T^4D_4$ ,                       | $F_4/T^4$ ,            | $\text{rank}(\Omega_1) = 4$ , $d = 24$ ; |
| 2) | $E_7/T^3A_1A_1A_1A_1$ ,                | $E_8/T^3A_1D_4$ ,                    | $F_4/T^3A_1$ ,         | $\text{rank}(\Omega_2) = 3$ , $d = 16$ ; |
| 3) | $E_7/T^3[A_1A_3]''$ ,                  | $E_8/T^3D_5$ ,                       | $F_4/T^3\tilde{A}_1$ , | $\text{rank}(\Omega_3) = 3$ , $d = 13$ ; |
| 4) | $E_6/T^3A_1A_2$ ,                      | $E_7/T^3A_4$ ,                       |                        | $\text{rank}(\Omega_4) = 3$ , $d = 10$ ; |
| 5) | $C_n/T^3A_{n_1-1}A_{n_2-1}A_{n_3-1}$ , | $D_{n+3}/T^3A_{n_1}A_{n_2}A_{n_3}$ , | $E_7/T^3D_4$ ,         | $\text{rank}(\Omega_5) = 3$ , $d = 9$ ;  |
| 6) |  | $D_{n+3}/T^3A_{n_1}A_{n_2}$ ,        | $E_6/T^3A_3$ ,         | $\text{rank}(\Omega_6) = 3$ , $d = 8$ .  |

(where  $n_i \geq 1$ ,  $\sum n_i = n$ ). Each of 1)-6) is one of  $\mathcal{F}(\Omega)$ .

Hence,  $\Omega_1$  is the exceptional root system of the type  $F_4$ , since  $F_4/T^4 \in \mathcal{F}(\Omega_1)$ .

*Proof.* The lines 1)-4) are contained in Tables 4 and 5 below, see § 5. Then there exist four sets  $\mathcal{F}(\Omega)$  such that every of them contains at least two exceptional flag manifolds. The lines 5) and 6) follows from Proposition 4.1 below. To complete the proof, we use Proposition 4.3.  $\square$

**Corollary.** *The condition of Theorem 6 holds for 15 irreducible  $T$ -root system  $\Omega$  of ranks  $\geq 2$ . In particular,  $|\mathcal{F}(\Omega)| \geq 2$  for the  $T$ -root system  $\Omega$  of each flag manifold  $E_\ell/T^k \cdot H$ , where  $k \geq 2$ , and  $H$  has a normal subgroup  $D_4$ ,  $D_5$ ,  $D_6$ , or  $E_6$ .*

We describe now the  $T$ -root systems 4)-6) of Theorem 6.

**Proposition 4.1.** *Flag manifolds  $E_6/T^3 \cdot A_3$  and  $E_7/T^3 \cdot D_4$  have  $T$ -root systems of types  $C_{3,2}$  and  $C_3$  respectively.*

**Proposition 4.2.** *The common  $T$ -root system  $\Omega$  of  $E_6/T^3 \cdot A_1 \cdot A_2$  and  $E_7/T^3 \cdot A_4$  is the union of classical root system of type  $B_3$  and two opposite vectors  $\mathbf{v}$ ,  $-\mathbf{v}$ , where  $\mathbf{v}$  is the sum of three short positive roots of  $B_3$ .*

<sup>1)</sup>In Theorem 6 we use Dynkin's notation  $[ ]''$ ,  $[ ]'$  for regular semisimple subgroups (subalgebras) of  $E_\ell$ ,  $\ell \in \{7, 8\}$ . Namely, we write  $[H]'$  if  $H \subset A_\ell$ , and  $[H]''$  otherwise. In 2), one can write  $[4A_1]''$  without  $4A_1$ , since  $4A_1 \supset [3A_1]''$ ; but  $E_7/T^3[4A_1]'$  is a non-flag manifold.

*Proof.* To find  $T$ -roots, we use each of two diagrams  $\begin{smallmatrix} 1 & 2 & 3 & 0 & 0 \\ & & & 0 & 0 \end{smallmatrix}$  and  $\begin{smallmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ & & & & 0 & 0 \end{smallmatrix}$ , where we indicate simple  $T$ -roots  $\alpha_i$ ,  $i = 1, 2, 3$ . The major  $T$ -root with respect to the basis  $\{\alpha_i\}$  is obviously  $\mathbf{v} = \alpha_1 + 2\alpha_2 + 3\alpha_3$ . It is easy to check that  $\Delta = \Omega \setminus \{\mathbf{v}, -\mathbf{v}\}$  is the root system of the type  $B_3$ , and  $\alpha_i$ ,  $i = 1, 2, 3$  form the standard basis of  $\Delta$  with the diagram  $1 \ 2 \Rightarrow 3$ .  $\square$

**Example.** Let  $\Omega$  be the  $T$ -root system of the flag manifold  $M = E_7/T^3 \cdot D_4$ . From Table 4 (§5) it follows that  $\Omega$  consists of  $2d = 18$  vectors in  $\mathbb{R}^3$ , and the convex hull of  $\Omega$  has 6 vertices. Obviously, we may write these vertices as  $\pm\alpha_1, \pm\alpha_2, \pm\alpha_3$ , so  $\text{Conv}(\Omega)$  is an octahedron. The others  $T$ -roots  $\gamma_i$ ,  $i = 1, \dots, 12$ , are the midpoints of twelve edges. Therefore  $\Omega$  is the root system of the type  $C_3$ . Let  $V_2$  be a 2-subspace of  $\mathbb{R}^3$  generated by midpoints of two intersecting edges, for definiteness,  $\frac{1}{2}(\alpha_1 - \alpha_2)$  and  $\frac{1}{2}(\alpha_3 - \alpha_2)$ . Then  $\Omega \cap V_2$  be a hexagonal root system of the type  $A_2$ . There is the root system of a flag submanifold  $M_2 = E_6/T^2 \cdot \text{Spin}(8) \subset M$ , since

$$T^2 \cdot \text{Spin}(8) \subset E_6 \subset E_7.$$

We describe the complexified isotropy representations of  $M$  and  $M_2$ . The  $T^3 \cdot \text{Spin}(8)$  module  $\mathbb{C} \otimes (\mathfrak{e}_7/\mathbb{R}^3 + \mathfrak{spin}(8))$  decomposes as a direct sum of 18 mutually non-equivalent irreducible submodules  $\mathfrak{m}_\omega$ ,  $\omega \in \Omega$ . Since  $M$  has a unique standard complex form (with the diagram  $\begin{smallmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ & & & 0 & 0 & 1 \end{smallmatrix}$ ), the Weyl group  $W(M)$  (i.e.,  $\text{Norm}(H)/H$ ,  $H = T^3 \cdot D_4$ ) acts transitively on the set of Weyl chambers of  $\Omega$ , and  $W(M) = W$ , the Weyl group of the root system  $C_3$ . So  $W(M)$  acts transitively on the subset of modules  $\mathfrak{m}_\gamma$  corresponding to midpoints of edges  $\gamma = \gamma_i$ ,  $i = 1, \dots, 12$ . Hence,  $\dim_{\mathbb{C}}(\mathfrak{m}_\gamma) = 1/6 \dim_{\mathbb{R}}(M_2) = 8$ . Similarly,  $W(M)$  acts transitively on the subset of the modules  $\mathfrak{m}_\alpha$  corresponding to the vertices  $\alpha \in \{\pm\alpha_i, i = 1, 2, 3\}$  of octahedron, and hence  $\dim_{\mathbb{C}}(\mathfrak{m}_\alpha) = 1$ , since

$$\dim_{\mathbb{R}}(M) - 12 \cdot 8 = (133 - 3 - 28) - 96 = 6.$$

We may regard the **real** isotropy module  $\mathfrak{e}_6/\mathbb{R}^2 + \mathfrak{spin}(8)$  of  $M_2$  as the direct sum of the complex vector and two complex semispinor modules of  $\text{Spin}(8)$  and the action of the subgroup  $S_3 = W_{A_2} = W(M_2)$  (permuting the  $\alpha_i$ ,  $i = 1, 2, 3$ ) of the group  $W$  as the Cartan's triality. A proof use symmetries of Dynkin's diagrams  $E_6$  and  $D_4$ . Another proof follows from the diagram

$$\text{SO}(10)/\text{SO}(2) \times \text{SO}(8) \longrightarrow E_6/T^2 \cdot \text{Spin}(8) \longrightarrow E_6/T^1 \cdot \text{Spin}(10) = (\mathbb{C} \otimes \mathbb{C}a)P^2.$$

$$\begin{smallmatrix} 1 & 0 & 0 & 0 \\ & & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 1 & 0 & 0 & 0 & 1 \\ & & & & 0 \end{smallmatrix} \quad \begin{smallmatrix} 0 & 0 & 0 & 0 & 1 \\ & & & & 0 \end{smallmatrix}$$

Thus, all  $\mathfrak{m}_\alpha$  are trivial  $\text{Spin}(8)$ -modules, and (given  $i < j \leq 3$ ) each  $\mathfrak{m}_\gamma$  is equivalent to  $\mathfrak{m}_{\frac{1}{2}(\alpha_i - \alpha_j)}$  as a  $\text{Spin}(8)$ -module if and only if  $\gamma = \frac{1}{2}(\pm\alpha_i \pm \alpha_j)$ .

**Note.** We give diagrams of the all standard complex forms of the spaces 1) and 3):

$$\begin{smallmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ & & & & & 0 \end{smallmatrix}, \quad \begin{smallmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ & & & & & & 0 \end{smallmatrix}, \quad 1 \Leftarrow 1 \ 1;$$

$$\begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ & & & & & 0 \end{smallmatrix}, \quad \begin{smallmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ & & & & & 0 \end{smallmatrix}, \quad \begin{smallmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & & & & & 0 \end{smallmatrix}, \quad \begin{smallmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ & & & & & & 0 \end{smallmatrix}, \quad 1 \ 0 \Leftarrow 1 \ 1, \quad 0 \ 1 \Leftarrow 1 \ 1.$$

This is a delicate case where exist two pairs of distinct flag manifolds of the same form  $G/H$  with 11 standard complex form in each pair. Namely,

$$E_7/T^4[A_1A_1A_1]'' \ (d = 24) \text{ and } E_7/T^4[A_1A_1A_1]' \ (d = 23)$$

have  $11 = 1 + 10$  standard complex forms. Further,

$$E_7/T^3[A_1A_3]'' \ (d = 13) \text{ and } E_7/T^3[A_1A_3]' \ (d = 12)$$

have also  $11 = 2 + 9$  standard complex forms. The second flag manifold in each pair has proportional  $T$ -roots  $\alpha$  and  $2\alpha$ . Remark also that

$$E_7/T^3[A_1A_3]'' \text{ and } E_7/T^3A_2A_2$$

have both  $T$ -root systems  $\Omega$  with  $d = \frac{1}{2}|\Omega| = 13$ . But the number of vertices of  $\text{Conv}(\Omega)$  equals 12 and 18 respectively.

#### 4.2. Description of $T$ -root systems of ranks $k \geq 2$ with $d \leq 10$ .

**Theorem 7.** *There are 16 irreducible  $T$ -root systems  $\Omega$  with  $\text{rank}(\Omega) \geq 2$  and  $d = \frac{1}{2}|\Omega| \leq 10$ , corresponding to flag manifolds of exceptional Lie groups:*

- some of  $T$ -root system of rank two, pictured in Table 1;
- the root system of the type  $C_3$ , and the  $T$ -root system of the type  $C_{3,2}$ , that is,  $C_3$  without two long opposite roots  $\alpha$  and  $-\alpha$  (Proposition 4.1);
- the system  $B_3 \cup \{\mathbf{v}, -\mathbf{v}\}$  of rank 3, described in Proposition 4.2.

#### 4.3. Exceptional and non-exceptional systems of $T$ -roots.

**Definition.** Let  $\Omega$  be an irreducible  $T$ -root system. We call  $\Omega$  **non-exceptional** if it corresponds to some flag manifold of a classical compact Lie group, and **exceptional**, otherwise.

E.g., flag manifolds  $E_6/T^3 \cdot A_3$  and  $E_7/T^3 \cdot D_4$  have non-exceptional  $T$ -root systems of types  $C_{3,2}$  and  $C_3$  respectively by Proposition 4.1.

From Table 4 in § 5 we obtain :

**Proposition 4.3.** *Let  $\Omega = \Omega_k$  be the system of  $T$ -roots of rank  $k$  corresponding to a flag manifold  $M$  with  $b_2(M) = k$  of a compact Lie group  $L = E_6, E_7, E_8$ , or  $F_4$ . Let  $M \neq E_6/T^3 \cdot A_3$ ,  $M \neq E_7/T^3 \cdot D_4$ , and  $k \geq 3$ . Then  $\Omega_k$  is exceptional.*

*Proof.* We may assume that  $k \in [3, 7]$ . Let  $d = \frac{1}{2}|\Omega_k|$ ,  $i = \frac{1}{2}|\omega \in \Omega_k : 2\omega \in \Omega_k|$ . Assume  $2v \leq 2d$  is the number of vertices of the convex hull of  $\Omega_k$ . From tables we obtain that there one of the following conditions holds.

- 1)  $k = 3$ ,  $d \in [k^2 - 1, k^2] = [8, 9]$ ,  $v < d$ ; then  $M = E_6/T^3 \cdot A_3$  or  $E_7/T^3 \cdot D_4$ ;
- 2)  $k \in [4, 6]$ ,  $d \in [k^2 - 1, k^2]$ ,  $v = d$  (there is a unique such system  $\Omega_k$  for each  $k \in [4, 6]$ , in particular, the root system  $E_6$ ,  $k = 6$ );
- 3)  $k \in [3, 5]$ ,  $d \in [k^2 + 1, k^2 + k]$ ,  $i \in [0, 1]$ ,  $i < d - k^2$  (there are  $6 - k$  such systems  $\Omega_k$ );
- 4)  $d > k^2 + k$ ; then  $\Omega_k$  is necessarily exceptional.

Consider the cases 2) and 3). Let  $\Omega_k$  is non-exceptional.

2) If  $d = k^2$ , then  $\Omega_k$  is the root system of the type  $B_k$  or  $C_k$ , and vertices of  $\text{Conv}(\Omega_k)$  are the long roots; hence  $v < d$ . If  $d = k^2 - 1$ , then  $\Omega_k$  is of the type  $C_{k,k-1}$ , so  $\Omega_k$  contains the subsystem  $C_{k-1}$ ; hence  $v < d$ . But  $v = d$ . The contradiction.

3) If  $d \in [k^2 + 1, k^2 + k]$ , then  $\Omega_k$  is the root system of the type  $BC_{k,i}$  (where  $BC_{k,k} = BC_k$ ), and  $i = d - k^2$ . But  $i < d - k^2$ . The contradiction. This prove Proposition 4.3 and completes the proof of Theorem 6.  $\square$



**Remark.** The complete list of non-exceptional irreducible  $T$ -root systems corresponding to flag manifolds of exceptional compact Lie groups is

$$A_1, BC_1, A_2, B_2, BC_{2,1}, BC_2, C_{3,2}, C_3,$$

as a consequence of Propositions 4.3 and 4.1.

## 5. TABLES OF $T$ -ROOT SYSTEMS AND THE CORRESPONDING FLAG MANIFOLDS OF EXCEPTIONAL SIMPLE LIE GROUPS

**5.1. Some invariants of  $T$ -root system.** Fix an irreducible  $T$ -root system  $\Omega$ , and the set  $\Omega^+$  of the positive  $T$ -roots. Assume

$$\Omega_V^+ := \{\omega \in \Omega^+ : 2\omega - \gamma \notin \Omega^+, \forall \gamma \in \Omega \cup (0), \gamma \neq \omega\}.$$

For every  $\omega \in \Omega$  we get  $g_\omega = \max\{k \in \mathbb{Z} : \frac{1}{k}\omega \in \Omega\}$ , and consider the following numbers:

$$d = \frac{1}{2} |\Omega| = |\Omega^+|, \quad v = |\Omega_V^+|,$$

$$c = \prod_{\omega \in \Omega^+} g_\omega, \quad w = \prod_{\omega \in \Omega_V^+} g_\omega.$$

It is clear that the numbers  $c, d, v, w$  are invariant under the action of the group  $\text{Aut}(\Omega)$ .

Note that  $2v$  is the number of the vertices of  $\text{Conv}(\Omega)$  by the following proposition:

**Proposition 5.1.** *The set of the vertices of the convex hull of  $\Omega$  coincides with*

$$\Omega_V := \{\omega \in \Omega : 2\omega - \gamma \notin \Omega \forall \gamma \in \Omega \cup (0), \gamma \neq \omega\},$$

*i.e., with  $\{+\omega, -\omega : \omega \in \Omega_V^+\}$ .*

Proposition 5.1 is not used in the classification of  $T$ -root systems.

We may clarify the meaning of  $c, w$  using the classification of the  $T$ -root systems of rank  $k = 2$ . Let

$$i(n) = |\omega \in \Omega^+ : n\omega \in \Omega^+|, \quad j(n) = |\omega \in \Omega^+ : n\omega \in \Omega_V^+|.$$

**Proposition 5.2.** *Let  $\Omega$  be an irreducible  $T$ -root system with rank  $k \geq 2$  of a flag manifold  $M \neq E_8/T^2 \cdot A_1 \cdot A_2 \cdot A_3$ . Then  $g_\omega \in \{1, 2, 3\}$  for all  $\omega \in \Omega$ , and*

$$c = 2^{i(2)} 3^{i(3)}, \quad w = 2^{j(2)} 3^{j(3)}.$$

**Proposition 5.3.** *Let  $\Omega$  be an irreducible  $T$ -root system of a rank  $k \geq 3$  of a flag manifold  $M \neq E_8/T^3 \cdot A_1 \cdot A_2 \cdot A_2$ . Then  $g_\omega \in \{1, 2\}$  for any  $\omega \in \Omega$ , and*

$$c = 2^{i(2)}, \quad w = 2^{j(2)}.$$

*Proof of Propositions 5.2 and 5.3.* Case  $k = 2$ . It follows from the classification of  $T$ -root systems of rank two, that  $m := \max g_\omega \leq 3$ . Cf. Tables 2 and 1. Case  $k \geq 3$ . Then  $m \leq 2$  by Proposition 3.4 (Corollary).  $\square$

To calculate  $c$  and  $w$  we use the following fact:

**Claim.**  $g_\omega$  is the g.c.d. of coefficients of  $\omega \in \Omega$  with respect to the basis of simple  $T$ -roots.

**5.2. Classification results. Tables.** In the following tables any  $T$ -root system  $\Omega$  is replaced by its invariants  $k, d, c, v, w$ , where  $k = \text{rank}(\Omega)$ .

Also as a result of classification of systems  $\Omega$  we state the following theorem.

Let  $M_i$ ,  $i = 1, 2$  are flag manifolds with  $b_2(M) \geq 2$  of exceptional compact simple Lie groups  $L_i$ ,  $i = 1, 2$ , and  $\Omega_i = \Omega_{M_i}$  the corresponding systems of  $T$ -roots.

**Theorem 8.** *The  $T$ -root systems  $\Omega_i = \Omega_{M_i}$ ,  $i = 1, 2$  with the same invariants  $k, d, c, v$  (or the same  $k, d, v, w$ ) are isomorphic.*

*Proof.* The proof is similar to the proof of Theorems 4 and 5. □

Propositions 5.1, 5.2 and 5.3 are not used in the classification of  $T$ -root systems and the proof of Theorem 8.

**Corollary.** *Theorem 8 remains valid for flag manifolds  $M_i$  with  $b_2(M_i) \geq 1$  of simple compact Lie groups  $L_i$ ,  $i = 1, 2$  (even classical).*

*Proof.* The proof is similar to the proof of Proposition 4.3. □

TABLE 4. Classification and invariants of T-root systems of ranks  $\geq 2$ 

**Notations.** 1) For flag manifolds:

- $f$  : the number of the standard complex forms of a flag manifold;
- $[1, 1], \dots, [0, 0]$  : for flag manifolds of  $E_\ell$ ,  $\ell = 7, 8$ ; we write  $[\ast, 1]$  (respectively  $[1, \ast]$ ) if the semisimple part of the isotropy subgroup is conjugate to a subgroup of the standard  $A_{\ell-1}$  (respectively  $E_{\ell-1}$ ).

2) For any  $T$ -root system  $\Omega$  :

- $d = \frac{1}{2} |\Omega| = |\Omega^+|$
- $c = \prod_{\omega \in \Omega^+} g_\omega$ , where  $g_\omega = \max\{k \in \mathbb{Z} : \frac{1}{k} \omega \in \Omega\}$
- $v = |\Omega_V^+|$ , where  $\Omega_V^+ := \{\omega \in \Omega^+ : 2\omega - \gamma \notin \Omega^+, \forall \gamma \in \Omega \cup (0), \gamma \neq \omega\}$ .
- $w = \prod_{\omega \in \Omega_V^+} g_\omega$

$$\text{rank}(\Omega) = 2$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^2 \bullet$ | $f$ | $E_7/T^2 \bullet$       | $f$ | $E_8/T^2 \bullet$       | $f$ | $F_4/T^2 \bullet$ | $f$ |
|-----|-----|-----|-----|-------------------|-----|-------------------------|-----|-------------------------|-----|-------------------|-----|
| 14  | 144 | 4   | 36  |                   |     |                         |     | $A_1 A_1 A_2 A_2$ [0 0] | 2   |                   |     |
| 12  | 48  | 4   | 8   |                   |     |                         |     | $A_1 A_2 A_3$ [1 0]     | 4   |                   |     |
| 11  | 8   | 4   | 4   |                   |     |                         |     | $A_1 A_1 A_4$ [0 0]     | 4   |                   |     |
| 10  | 6   | 4   | 3   |                   |     |                         |     | $A_2 A_4$ [1 1]         | 4   |                   |     |
| 10  | 4   | 4   | 1   |                   |     |                         |     | $A_3 A_3$ [0 1]         | 2   |                   |     |
| 9   | 8   | 3   | 8   |                   |     | $A_1 A_1 A_1 A_2$ [0 0] | 1   | $A_2 D_4$ [0 0]         | 1   | $A_2$             | 1   |
| 9   | 12  | 4   | 6   |                   |     |                         |     | $A_1 A_5$ [1 1]         | 3   |                   |     |
| 8   | 4   | 3   | 2   |                   |     | $A_1 A_1 A_3$ [0 0]     | 3   | $A_1 D_5$ [1 0]         | 3   | $A_1 \tilde{A}_1$ | 3   |
| 8   | 6   | 3   | 3   |                   |     | $A_1 A_2 A_2$ [1 0]     | 3   |                         |     |                   |     |
| 7   | 2   | 4   | 2   |                   |     | $A_2 A_3$ [0 1]         | 3   | $A_6$ [1 1]             | 3   |                   |     |
| 6   | 1   | 3   | 1   | $A_2 A_2$         | 1   | $A_5'$ [0 0]            | 1   | $E_6$ [1 0]             | 1   | $\tilde{A}_2$     | 1   |
| 6   | 2   | 3   | 2   | $A_1 A_1 A_2$     | 5   | $A_1 A_4$ [1 1]         | 5   |                         |     |                   |     |
| 6   | 4   | 2   | 4   |                   |     | $A_1 D_4$ [0 0]         | 1   | $D_6$ [1 0]             | 1   | $B_2$             | 1   |
| 5   | 2   | 3   | 2   | $A_1 A_3$         | 4   | $A_5'$ [1 1]            | 2   |                         |     |                   |     |
| 4   | 1   | 2   | 1   | $A_4$             | 4   | $D_5$ [1 0]             | 2   |                         |     |                   |     |
| 3   | 1   | 3   | 1   | $D_4$             | 1   |                         |     |                         |     |                   |     |

Note. We have  $g_\omega \in \{1, 2, 3\}$  for all  $\omega \in \Omega$ , except for two  $T$ -roots  $+\omega, -\omega$  with  $g_{\pm\omega} = 4$  in the case  $d = 12$ ,

$$c = 48 = 2^2 \cdot 3 \cdot 4.$$

$$\text{rank}(\Omega) = 3$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^3\bullet$ | $f$ | $E_7/T^3\bullet$ | $f$      | $E_8/T^3\bullet$ | $f$      | $F_4/T^3\bullet$ | $f$ |                   |   |
|-----|-----|-----|-----|------------------|-----|------------------|----------|------------------|----------|------------------|-----|-------------------|---|
| 23  | 16  | 10  | 16  |                  |     |                  |          | $A_1A_1A_1A_2$   | $[1\ 0]$ | 8                |     |                   |   |
| 21  | 6   | 10  | 3   |                  |     |                  |          | $A_1A_2A_2$      | $[1\ 1]$ | 8                |     |                   |   |
| 20  | 8   | 10  | 4   |                  |     |                  |          | $A_1A_1A_3$      | $[1\ 1]$ | 10               |     |                   |   |
| 18  | 2   | 11  | 2   |                  |     |                  |          | $A_2A_3$         | $[1\ 1]$ | 10               |     |                   |   |
| 17  | 2   | 10  | 2   |                  |     |                  |          | $A_1A_4$         | $[1\ 1]$ | 12               |     |                   |   |
| 16  | 8   | 7   | 8   |                  |     | $4A_1$           | $[0\ 0]$ | 2                | $A_1D_4$ | $[1\ 0]$         | 2   | $A_1$             | 2 |
| 14  | 2   | 10  | 2   |                  |     |                  |          | $A_5$            | $[1\ 1]$ | 4                |     |                   |   |
| 14  | 2   | 8   | 2   |                  |     | $A_1A_1A_2$      | $[1\ 1]$ | 12               |          |                  |     |                   |   |
| 13  | 1   | 9   | 1   |                  |     | $A_2A_2$         | $[1\ 1]$ | 4                |          |                  |     |                   |   |
| 13  | 1   | 6   | 1   |                  |     | $[A_1A_3]''$     | $[0\ 0]$ | 2                | $D_5$    | $[1\ 0]$         | 2   | $\widetilde{A}_1$ | 2 |
| 12  | 2   | 8   | 2   |                  |     | $[A_1A_3]'$      | $[1\ 1]$ | 9                |          |                  |     |                   |   |
| 11  | 2   | 7   | 2   | $A_1A_1A_1$      | 5   |                  |          |                  |          |                  |     |                   |   |
| 10  | 1   | 7   | 1   | $A_1A_2$         | 10  | $A_4$            | $[1\ 1]$ | 5                |          |                  |     |                   |   |
| 9   | 1   | 3   | 1   |                  |     | $D_4$            | $[1\ 0]$ | 1                |          |                  |     |                   |   |
| 8   | 1   | 6   | 1   | $A_3$            | 5   |                  |          |                  |          |                  |     |                   |   |

Note. We have  $g_\omega \in \{1, 2\}$  for all  $\omega \in \Omega$ , except for two  $T$ -roots  $+\omega, -\omega$  with  $g_{\pm\omega} = 3$  in the case  $d = 21, c = 6$ .

$$\text{rank}(\Omega) = 4$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^4 \bullet$ | $f$ | $E_7/T^4 \bullet$ | $f$      | $E_8/T^4 \bullet$ | $f$      | $F_4/T^4 \bullet$ | $f$ |         |   |
|-----|-----|-----|-----|-------------------|-----|-------------------|----------|-------------------|----------|-------------------|-----|---------|---|
| 36  | 16  | 20  | 16  |                   |     |                   |          | $4A_1$            | $[1\ 1]$ | 7                 |     |         |   |
| 33  | 2   | 21  | 2   |                   |     |                   |          | $A_1A_1A_2$       | $[1\ 1]$ | 28                |     |         |   |
| 30  | 1   | 24  | 1   |                   |     |                   |          | $A_2A_2$          | $[1\ 1]$ | 8                 |     |         |   |
| 29  | 2   | 21  | 2   |                   |     |                   |          | $A_1A_3$          | $[1\ 1]$ | 20                |     |         |   |
| 25  | 1   | 20  | 1   |                   |     |                   |          | $A_4$             | $[1\ 1]$ | 6                 |     |         |   |
| 24  | 1   | 12  | 1   |                   |     | $A_1A_1A_1$       | $[0\ 0]$ | 1                 | $D_4$    | $[1\ 0]$          | 1   | $\{e\}$ | 1 |
| 23  | 2   | 16  | 2   |                   |     | $A_1A_1A_1$       | $[1\ 1]$ | 10                |          |                   |     |         |   |
| 21  | 1   | 17  | 1   |                   |     | $A_1A_2$          | $[1\ 1]$ | 18                |          |                   |     |         |   |
| 18  | 1   | 15  | 1   |                   |     | $A_3$             | $[1\ 1]$ | 6                 |          |                   |     |         |   |
| 17  | 1   | 14  | 1   | $A_1A_1$          | 10  |                   |          |                   |          |                   |     |         |   |
| 15  | 1   | 15  | 1   | $A_2$             | 5   |                   |          |                   |          |                   |     |         |   |

$$\text{rank}(\Omega) = 5$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^5 \bullet$ | $f$ | $E_7/T^5 \bullet$ | $f$         | $E_8/T^5 \bullet$ | $f$         |
|-----|-----|-----|-----|-------------------|-----|-------------------|-------------|-------------------|-------------|
| 50  | 2   | 37  | 2   |                   |     |                   |             | $A_1 A_1 A_1$     | $[1\ 1]$ 21 |
| 46  | 1   | 40  | 1   |                   |     |                   |             | $A_1 A_2$         | $[1\ 1]$ 28 |
| 41  | 1   | 36  | 1   |                   |     |                   |             | $A_3$             | $[1\ 1]$ 7  |
| 33  | 1   | 29  | 1   |                   |     | $A_1 A_1$         | $[1\ 1]$ 15 |                   |             |
| 30  | 1   | 30  | 1   |                   |     | $A_2$             | $[1\ 1]$ 6  |                   |             |
| 25  | 1   | 25  | 1   | $A_1$             | 6   |                   |             |                   |             |

$$\text{rank}(\Omega) = 6$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^6 \bullet$ | $f$ | $E_7/T^6 \bullet$ | $f$        | $E_8/T^6 \bullet$ | $f$         |
|-----|-----|-----|-----|-------------------|-----|-------------------|------------|-------------------|-------------|
| 68  | 1   | 62  | 1   |                   |     |                   |            | $A_1 A_1$         | $[1\ 1]$ 21 |
| 63  | 1   | 63  | 1   |                   |     |                   |            | $A_2$             | $[1\ 1]$ 7  |
| 46  | 1   | 46  | 1   |                   |     | $A_1$             | $[1\ 1]$ 7 |                   |             |
| 36  | 1   | 36  | 1   | $\{e\}$           | 1   |                   |            |                   |             |

$$\text{rank}(\Omega) = 7$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^7 \bullet$ | $f$ | $E_7/T^7 \bullet$ | $f$        | $E_8/T^7 \bullet$ | $f$        |
|-----|-----|-----|-----|-------------------|-----|-------------------|------------|-------------------|------------|
| 91  | 1   | 91  | 1   |                   |     |                   |            | $A_1$             | $[1\ 1]$ 8 |
| 63  | 1   | 63  | 1   |                   |     | $\{e\}$           | $[1\ 1]$ 1 |                   |            |

$$\text{rank}(\Omega) = 8$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^8 \bullet$ | $f$ | $E_7/T^8 \bullet$ | $f$ | $E_8/T^8 \bullet$ | $f$        |
|-----|-----|-----|-----|-------------------|-----|-------------------|-----|-------------------|------------|
| 120 | 1   | 120 | 1   |                   |     |                   |     | $\{e\}$           | $[1\ 1]$ 1 |

TABLE 5. Consistent numberings of simple T-roots

Consistent numberings of simple T-roots (in each line). We skip 5 of 10 diagrams of the type  $E_6$ . These 5 diagrams may be obtained by symmetry of the Dynkin graph.

| $k$ | $d$ | $c$ | $v$ | $w$ | $E_6/T^k H$   | $f$ | $E_7/T^k H$   | $f$ | $E_8/T^k H$   | $f$ | $F_4/T^k H$   | $f$ |
|-----|-----|-----|-----|-----|---|-----|---|-----|---|-----|---|-----|
| 4   | 24  | 1   | 12  | 1   |   |     | $E_7/T^4[A_1 A_1 A_1]''$<br>0 1 0 2 3 4<br>0  | 1   | $E_8/T^4 D_4$<br>4 3 2 0 0 0 1<br>0   | 1   | $F_4/T^4$<br>1 2 $\Leftarrow$ 3 4   | 1   |
| 3   | 16  | 8   | 7   | 8   |   |     | $E_7/T^3 A_1 A_1 A_1 A_1$<br>0 1 0 2 3 0<br>0<br>0 1 0 2 0 3<br>0   | 2   | $E_8/T^3 A_1 D_4$<br>0 3 2 0 0 0 1<br>0<br>3 0 2 0 0 0 1<br>0                       | 2   | $F_4/T^3 A_1$<br>1 2 $\Leftarrow$ 3 0<br>1 2 $\Leftarrow$ 0 3                                     | 2   |
| 3   | 13  | 1   | 6   | 1   |   |     | $E_7/T^3[A_1 A_3]''$<br>0 1 0 0 2 3<br>0<br>0 0 0 1 2 3<br>0  | 2   | $E_8/T^3 D_5$<br>3 2 0 0 0 0 1<br>0<br>3 2 1 0 0 0 0<br>0                           | 2   | $F_4/T^3 \tilde{A}_1$<br>1 0 $\Leftarrow$ 2 3<br>0 1 $\Leftarrow$ 2 3                             | 2   |
| 3   | 10  | 1   | 7   | 1   | $E_6/T^3 A_1 A_2$<br>1 2 3 0 0<br>0<br>1 2 0 3 0<br>0<br>1 0 2 0 0<br>3<br>1 0 0 2 0<br>3<br>0 0 3 2 0<br>1 | 10  | $E_7/T^3 A_4$<br>1 2 3 0 0 0<br>0<br>1 2 0 0 0 0<br>3<br>1 0 0 0 2 3<br>0<br>1 0 0 0 0 3<br>2<br>0 0 0 0 3 1<br>2 | 5   |   |     |   |     |
| 2   | 8   | 4   | 3   | 2   |   |     | $E_7/T^2 A_1 A_1 A_3$<br>0 1 0 0 2 0<br>0<br>0 0 0 1 2 0<br>0<br>0 0 0 1 0 2<br>0                                 | 3   | $E_8/T^2 A_1 D_5$<br>0 2 0 0 0 0 1<br>0<br>0 2 1 0 0 0 0<br>0<br>2 0 1 0 0 0 0<br>0 | 3   | $F_4/T^2 A_1 \tilde{A}_1$<br>1 0 $\Leftarrow$ 2 0<br>0 1 $\Leftarrow$ 2 0<br>0 1 $\Leftarrow$ 0 2 | 3   |
| 2   | 7   | 2   | 4   | 2   |   |     | $E_7/T^2 A_2 A_3$<br>0 0 1 0 0 2<br>0<br>0 0 1 0 0 0<br>2<br>0 0 0 1 0 0<br>2                                     | 3   | $E_8/T^2 A_6$<br>2 0 0 0 0 0 0<br>1<br>0 0 0 0 0 0 2<br>1<br>0 0 0 0 0 1 2<br>0     | 3   |   |     |

TABLE 6. The same table as 4 with diagrams of some extremal complex forms

The same, as Table 4, with diagrams of some standard complex forms of flag manifolds. The positive labels (on each diagram) are coefficients of the major  $T$ -root. We skip only the diagrams with coprime labels. For  $\text{rank}(\Omega) > 5$  all diagrams are skipped.

$$\text{rank}(\Omega) = 2$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^2 \bullet$                             | $f$ | $E_7/T^2 \bullet$                     | $f$ | $E_8/T^2 \bullet$   | $f$ | $F_4/T^2 \bullet$                         | $f$ |
|-----|-----|-----|-----|---|-----|---------------------------------------|-----|---|-----|---|-----|
| 14  | 144 | 4   | 36  |   |     |                                       |     | $A_1 A_1 A_2 A_2$<br>0 3 0 0 6 0 0<br>0<br>0 0 4 0 6 0 0<br>0 | 2   |   |     |
| 12  | 48  | 4   | 8   |   |     |                                       |     | $A_1 A_2 A_3$<br>2 0 0 0 6 0 0<br>0<br>0 0 4 0 0 4 0<br>0     | 4   |   |     |
| 11  | 8   | 4   | 4   |   |     |                                       |     | $A_1 A_1 A_4$<br>0 0 0 0 6 4 0<br>0<br>0 0 0 0 6 0 2<br>0     | 4   |   |     |
| 10  | 6   | 4   | 3   |   |     |                                       |     | $A_2 A_4$<br>0 0 0 0 6 0 0<br>3                               | 4   |   |     |
| 10  | 4   | 4   | 1   |   |     |                                       |     | $A_3 A_3$   | 2   |   |     |
| 9   | 8   | 3   | 8   |   |     | $A_1 A_1 A_1 A_2$<br>0 2 0 4 0 0<br>0 | 1   | $A_2 D_4$<br>0 0 4 0 0 0 2<br>0                               | 1   | $A_2$<br>2 4 $\Leftarrow$ 0 0             | 1   |
| 9   | 12  | 4   | 6   |   |     |                                       |     | $A_1 A_5$<br>2 0 0 0 0 4 0<br>0<br>0 3 0 0 0 0 0<br>3         | 3   |   |     |
| 8   | 4   | 3   | 2   |   |     | $A_1 A_1 A_3$<br>0 0 0 4 0 2<br>0     | 3   | $A_1 D_5$<br>2 0 4 0 0 0 0<br>0                               | 3   | $A_1 \tilde{A}_1$<br>0 4 $\Leftarrow$ 0 2 | 3   |
| 8   | 6   | 3   | 3   |   |     | $A_1 A_2 A_2$<br>0 0 3 0 3 0<br>0     | 3   |   |     |   |     |
| 7   | 2   | 4   | 2   |   |     | $A_2 A_3$<br>0 0 0 4 0 0<br>2         | 3   | $A_6$<br>0 0 0 0 0 4 2<br>0                                   | 3   |   |     |
| 6   | 2   | 3   | 2   | $A_1 A_1 A_2$<br>0 2 0 2 0<br>0               | 5   | $A_1 A_4$<br>0 2 0 0 0 0<br>2         | 5   |   |     |   |     |
| 6   | 4   | 2   | 4   |   |     | $A_1 D_4$<br>0 2 0 0 0 2<br>0         | 1   | $D_6$<br>2 0 0 0 0 0 2<br>0                                   | 1   | $B_2$<br>2 0 $\Leftarrow$ 0 2             | 1   |
| 6   | 1   | 3   | 1   | $A_2 A_2$                                     | 1   | $A_5''$                               | 1   | $E_6$   | 1   | $\tilde{A}_2$                             | 1   |
| 5   | 2   | 3   | 2   | $A_1 A_3$<br>0 2 0 0 0<br>2<br>0 0 0 2 0<br>2 | 4   | $A_5'$<br>0 0 0 0 0 2<br>2            | 2   |   |     |   |     |
| 4   | 1   | 2   | 1   | $A_4$   | 4   | $D_5$                                 | 2   |   |     |   |     |
| 3   | 1   | 3   | 1   | $D_4$   | 1   |                                       |     |   |     |   |     |

$$\text{rank}(\Omega) = 3$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^3 \bullet$                           | $f$ | $E_7/T^3 \bullet$  | $f$ | $E_8/T^3 \bullet$   | $f$ | $F_4/T^3 \bullet$                 | $f$ |
|-----|-----|-----|-----|---|-----|--|-----|---|-----|-----------------------------------|-----|
| 23  | 16  | 10  | 16  |   |     |  |     | $A_1 A_1 A_1 A_2$<br>$2 \ 0 \ 4 \ 0 \ 6 \ 0 \ 0$<br>$0$<br>$0 \ 0 \ 4 \ 0 \ 6 \ 4 \ 0$<br>$0$<br>$0 \ 0 \ 4 \ 0 \ 6 \ 0 \ 2$<br>$0$ | 8   |                                   |     |
| 21  | 6   | 10  | 3   |   |     |  |     | $A_1 A_2 A_2$<br>$0 \ 3 \ 0 \ 0 \ 6 \ 0 \ 0$<br>$3$   | 8   |                                   |     |
| 20  | 8   | 10  | 4   |   |     |  |     | $A_1 A_1 A_3$<br>$2 \ 0 \ 4 \ 0 \ 0 \ 4 \ 0$<br>$0$<br>$2 \ 0 \ 0 \ 0 \ 6 \ 4 \ 0$<br>$0$<br>$2 \ 0 \ 0 \ 0 \ 6 \ 0 \ 2$<br>$0$     | 10  |                                   |     |
| 18  | 2   | 11  | 2   |   |     |  |     | $A_2 A_3$<br>$0 \ 0 \ 4 \ 0 \ 0 \ 4 \ 2$<br>$0$   | 10  |                                   |     |
| 17  | 2   | 10  | 2   |   |     |  |     | $A_1 A_4$<br>$0 \ 0 \ 0 \ 0 \ 6 \ 4 \ 2$<br>$0$   | 12  |                                   |     |
| 16  | 8   | 7   | 8   |   |     | $A_1 A_1 A_1 A_1$<br>$0 \ 2 \ 0 \ 4 \ 0 \ 2$<br>$0$                              | 2   | $A_1 D_4$<br>$2 \ 0 \ 4 \ 0 \ 0 \ 0 \ 2$<br>$0$   | 2   | $A_1$<br>$2 \ 4 \Leftarrow 0 \ 2$ | 2   |
| 14  | 2   | 8   | 2   |   |     | $A_1 A_1 A_2$<br>$0 \ 2 \ 0 \ 4 \ 0 \ 0$<br>$2$                                  | 12  |   |     |                                   |     |
| 14  | 2   | 10  | 2   |   |     |  |     | $A_5$<br>$2 \ 0 \ 0 \ 0 \ 0 \ 4 \ 2$<br>$0$   | 4   |                                   |     |
| 13  | 1   | 9   | 1   |   |     | $A_2 A_2$  | 4   |   |     |                                   |     |
| 13  | 1   | 6   | 1   |   |     | $[A_1 A_3]''$  | 2   | $D_5$   | 2   | $\tilde{A}_1$                     | 2   |
| 12  | 2   | 8   | 2   |   |     | $[A_1 A_3]'$<br>$0 \ 2 \ 0 \ 0 \ 0 \ 2$<br>$2$<br>$0 \ 0 \ 0 \ 4 \ 0 \ 2$<br>$2$ | 9   |   |     |                                   |     |
| 11  | 2   | 7   | 2   | $A_1 A_1 A_1$<br>$0 \ 2 \ 0 \ 2 \ 0$<br>$2$ | 5   |  |     |   |     |                                   |     |
| 10  | 1   | 7   | 1   | $A_1 A_2$                                   | 10  | $A_4$  | 5   |   |     |                                   |     |
| 9   | 1   | 3   | 1   |   |     | $D_4$  | 1   |   |     |                                   |     |
| 8   | 1   | 6   | 1   | $A_3$                                       | 5   |  |     |   |     |                                   |     |



$$\text{rank}(\Omega) = 4$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^4 \bullet$ | $f$ | $E_7/T^4 \bullet$                    | $f$ | $E_8/T^4 \bullet$   | $f$ | $F_4/T^4 \bullet$ | $f$ |
|-----|-----|-----|-----|-------------------|-----|--------------------------------------|-----|---|-----|-------------------|-----|
| 36  | 16  | 20  | 16  |                   |     |                                      |     | $A_1 A_1 A_1 A_1$<br>2 0 4 0 6 4 0<br>0<br>2 0 4 0 6 0 2<br>0 | 7   |                   |     |
| 33  | 2   | 21  | 2   |                   |     |                                      |     | $A_1 A_1 A_2$<br>0 0 4 0 6 4 2<br>0                           | 28  |                   |     |
| 30  | 1   | 24  | 1   |                   |     |                                      |     | $A_2 A_2$   | 8   |                   |     |
| 29  | 2   | 21  | 2   |                   |     |                                      |     | $A_1 A_3$<br>2 0 4 0 0 4 2<br>0<br>2 0 0 0 6 4 2<br>0         | 20  |                   |     |
| 25  | 1   | 20  | 1   |                   |     |                                      |     | $A_4$   | 6   |                   |     |
| 24  | 1   | 12  | 1   |                   |     | $[A_1 A_1 A_1]''$                    | 1   | $D_4$   | 1   | $\{e\}$           | 1   |
| 23  | 2   | 16  | 2   |                   |     | $[A_1 A_1 A_1]'$<br>0 2 0 4 0 2<br>2 | 10  |   |     |                   |     |
| 21  | 1   | 17  | 1   |                   |     | $A_1 A_2$                            | 18  |   |     |                   |     |
| 18  | 1   | 15  | 1   |                   |     | $A_3$                                | 6   |   |     |                   |     |
| 17  | 1   | 14  | 1   | $A_1 A_1$         | 10  |                                      |     |   |     |                   |     |
| 15  | 1   | 15  | 1   | $A_2$             | 5   |                                      |     |   |     |                   |     |

$$\text{rank}(\Omega) = 5$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^5 \bullet$ | $f$ | $E_7/T^5 \bullet$ | $f$ | $E_8/T^5 \bullet$                   | $f$ |
|-----|-----|-----|-----|-------------------|-----|-------------------|-----|-------------------------------------|-----|
| 50  | 2   | 37  | 2   |                   |     |                   |     | $A_1 A_1 A_1$<br>2 0 4 0 6 4 2<br>0 | 21  |
| 46  | 1   | 40  | 1   |                   |     |                   |     | $A_1 A_2$                           | 28  |
| 41  | 1   | 36  | 1   |                   |     |                   |     | $A_3$                               | 7   |
| 33  | 1   | 29  | 1   |                   |     | $A_1 A_1$         | 15  |                                     |     |
| 30  | 1   | 30  | 1   |                   |     | $A_2$             | 6   |                                     |     |
| 25  | 1   | 25  | 1   | $A_1$             | 6   |                   |     |                                     |     |

$$\text{rank}(\Omega) = 6$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^6 \bullet$ | $f$ | $E_7/T^6 \bullet$ | $f$ | $E_8/T^6 \bullet$ | $f$ |
|-----|-----|-----|-----|-------------------|-----|-------------------|-----|-------------------|-----|
| 68  | 1   | 62  | 1   |                   |     |                   |     | $A_1 A_1$         | 21  |
| 63  | 1   | 63  | 1   |                   |     |                   |     | $A_2$             | 7   |
| 46  | 1   | 46  | 1   |                   |     | $A_1$             | 7   |                   |     |
| 36  | 1   | 36  | 1   | $\{e\}$           | 1   |                   |     |                   |     |

$$\text{rank}(\Omega) = 7$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^7 \bullet$ | $f$ | $E_7/T^7 \bullet$ | $f$ | $E_8/T^7 \bullet$ | $f$ |
|-----|-----|-----|-----|-------------------|-----|-------------------|-----|-------------------|-----|
| 91  | 1   | 91  | 1   |                   |     |                   |     | $A_1$             | 8   |
| 63  | 1   | 63  | 1   |                   |     | $\{e\}$           | 1   |                   |     |

$$\text{rank}(\Omega) = 8$$

| $d$ | $c$ | $v$ | $w$ | $E_6/T^8 \bullet$ | $f$ | $E_7/T^8 \bullet$ | $f$ | $E_8/T^8 \bullet$ | $f$ |
|-----|-----|-----|-----|-------------------|-----|-------------------|-----|-------------------|-----|
| 120 | 1   | 120 | 1   |                   |     |                   |     | $\{e\}$           | 1   |

## REFERENCES

- [1] D. V. Alekseevsky and A. M. Perelomov, Invariant Kähler-Einstein metrics on compact homogeneous spaces, *Funktsional. Anal. i Prilozhen.* 20:3 (1986), 1-16; English transl., *Funct. Anal. Appl.* 20:3 (1986), 171-182.
- [2] N. Bourbaki. *Éléments de mathématique. Groupes et algèbres de Lie. Chap. IV–VI.* Hermann. Paris, 1968.
- [3] E. B. Dynkin, Semi-simple subalgebras of semi-simple Lie algebras, *Mat. Sb.* 72:2, 349–462 (1952). English translation: *Amer. Math. Soc. Transl.*, 2, 111–244 (1957).
- [4] Э.Б. Винберг, А.Л. Онищик. Семинар по группам Ли и алгебраическим группам. E.B. Vinberg and A.L. Onishchik, *Lie Groups and Algebraic Groups.* Springer-Verlag, 1990, 328p.
- [5] E.B. Vinberg, V.V. Gorbatshevich and A.L. Onishchik, *Structure of Lie groups and Lie algebras.* Encyclopaedia of Mathematical Sciences, 41, Springer-Verlag, 1994, 1-244.
- [6] M.M.Graev. On the number of invariant Einstein metrics on a compact homogeneous space, Newton polytopes and contractions of Lie algebras. *Int. J. G. M. Mod. Phys.* Vol. 3, Nos. 5 & 6 (2006) 1047-1075
- [7] Andreas Arvanitoyeorgos, Ioannis Chrysikos, Yusuke Sakane, Homogeneous Einstein metrics on generalized flag manifolds with  $G_2$ -type  $t$ -roots. *Proc. 3rd International Colloquium on Differential Geometry and its Related Fields.* Veliko Tarnovo, September 3–7, 2012.
- [8] Michail M. Graev. On invariant Einstein metrics on Kahler homogeneous spaces  $SU_4/T^3$ ,  $G_2/T^2$ ,  $E_6/T^2(A_2)^2$ ,  $E_7/T^2A_5$ ,  $E_8/T^2E_6$ ,  $F_4/T^2A_2$  arXiv:1111.0639 (Russian, English abstract).

M.M.GRAEV, SCIENTIFIC RESEARCH INSTITUTE FOR SYSTEM STUDIES OF RUSSIAN ACADEMY OF SCIENCES, NAKHIMOVSKY PROSP., 36, KORPUS 1, MOSCOW, 117218, RUSSIA  
*E-mail address:* mmgraev@gmail.com